Weak Cosmic Censorship for the Einstein Scalar Field Equations in Spherical Symmetry

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Abstract

The current essay is an expository work presenting Christodoulou's proof in [6] of the Weak Cosmic Censorship for the Einstein scalar field equations in spherical symmetry. The key idea of the proof is that the blue-shift effect represents an instability mechanism, forcing solutions with naked singularities to be unstable. We also briefly illustrate the other results that enter the proof of the Weak Cosmic Censorship, as in [3] – [4] and [8] – [10].

1 Introduction

The fundamental object of general relativity is a spacetime (M, g), together with various matter models defined on it, which satisfies the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu},$$

coupled with the corresponding matter equations. Here $T_{\mu\nu}$ represents the stress-energy-momentum tensor. Given matter that arises from a Lagrangian, there is a canonical way to obtain $T_{\mu\nu}$. For example, if the matter Lagrangian is that of a massive scalar field ϕ, then (M, g, ϕ) need to satisfy the Einstein-Klein-Gordon system of equations. Similarly, we can consider as matter models an electromagnetic field to obtain the Einstein-Maxwell system or a barotropic relativistic perfect fluid to obtain the Einstein-Euler system.

One of the most remarkable features of general relativity is that the theory is non-trivial even in the absence of matter. Thus, considering the trivial stress-energy-momentum tensor, we obtain the vacuum Einstein equations:

$$R_{\mu\nu} = 0,$$

which admit curved solutions.

In the above covariant form, it is unclear what "type" of PDEs the Einstein vacuum equations are. However, in a suitable gauge the equations are manifestly hyperbolic, so we can study the initial value problem. Initial data for the Einstein vacuum equations consist of a triple (Σ, h, k), where (Σ, h) is a $3-$dimensional Riemannian manifold, and k is a symmetric covariant $2-$tensor. In order to have that Σ is (isometric to) a spacelike hypersurface in a spacetime (M, g), with induced metric h and second fundamental form k, we note that h and k need to satisfy the Gauss-Codazzi equations. We refer to these as the *constraint equations*. We cite the following result from [1]:

Theorem 1. *Consider initial data (Σ, h, k) for the Einstein vacuum equations, satisfying the constraint equations. There exists, up to isometry, a unique globally hyperbolic spacetime (M, g) solving the Einstein vacuum equations, with Cauchy hypersurface Σ such that the induced metric and second fundamental forms on Σ are h and k, and that is the maximal spacetime satisfying these properties. We call this spacetime the maximal globally hyperbolic Cauchy development of the initial data (Σ, h, k).*

We remark that similar results hold when we couple the Einstein equations with matter satisfying hyperbolic PDEs.

Following the above result, it is natural to ask whether this maximal spacetime is complete. Specific examples such as the Schwarzschild solutions do not satisfy this property, but one may think this is just an artefact of spherical symmetry. It turns out we can infer that a spacetime is incomplete under assumptions that are stable to perturbation. This is the famous Incompleteness Theorem of Penrose, proved in [11], for which he received the Nobel Prize ([14]).

Theorem 2. *Consider (M, g) to be a connected globally hyperbolic spacetime with a non-compact Cauchy hypersurface Σ. Assume that the spacetime satisfies the null energy condition, i.e. $R(v, v) \geq 0,\ \forall\ v$ null. If the spacetime also contains a closed trapped surface, then it is future geodesically incomplete.*

We note that the above theorem does not say anything about the reason for incompleteness: an incomplete geodesic could either reach a curvature singularity, or it could exit the globally hyperbolic domain in finite proper time. We remark that it also assumes the existence of a trapped surface in the initial data, an issue that we will address in section 4.

In the case of a Schwarzschild black hole with positive mass parameter, the curvature singularity is "hidden" behind the event horizon. However, in general there is also the possibility that a curvature singularity is visible to observers outside of the black hole, in particular it is in the causal past of far away observers. Such a singularity is called a *naked singularity*. In [12], Penrose conjectured that naked singularities do not occur generically:

Conjecture 1 (Weak Cosmic Censorship)**.** *For generic asymptotically flat initial data for the Einstein vacuum equations, the maximal globally hyperbolic future Cauchy development does not contain a naked singularity.*

Starting from the idea that a naked singularity is visible to far away observers, so they cannot exist for infinite time, the Weak Cosmic Censorship conjecture can be restated as:

Conjecture 2 (Weak Cosmic Censorship, [7]). *For generic asymptotically flat initial data for the Einstein vacuum equations, the maximal globally hyperbolic future Cauchy development has a complete future null infinity* $\mathcal{I}^+$.

We now briefly explain why the genericity assumption is necessary in the above conjectures. The textbook example of a naked singularity is that of negative mass Schwarzschild, but in this case the initial data is not asymptotically flat. However, Christodoulou constructed spherically symmetric examples of asymptotically flat spacetimes with naked singularities for dust ([2]) and for a self-gravitating scalar field ([5]). More recently, Rodnianski and Shlapentokh-Rothman constructed examples of asymptotically flat spacetimes with naked singularities for the Einstein vacuum equations, which are not spherically symmetric ([13]).

The Weak Cosmic Censorship conjecture is one of the main open problems in general relativity. Part of the difficulty lies in the fact that the Einstein vacuum equations are a nonlinear system of 10 equations in $1+3$ dimensions. In view of this we make simplifying assumptions, so we consider solutions satisfying certain symmetries. The most natural such assumption is that of spherical symmetry, which we make precise now:

Definition 1. *A spacetime* (M, g) *is said to be spherically symmetric if its isometry group contains a subgroup isomorphic to* $SO(3)$*, with orbits* $2-$*spheres.*

However, Birkhoff's theorem states that vacuum spherically symmetric spacetimes are locally isometric to the Schwarzschild solution (with parameter $M > 0,\ M = 0$, or $M < 0$). A similar rigidity result holds for charged spherically symmetric spacetimes. The most simple non-singular matter model for which spherically symmetric solutions have dynamics is that of a massless scalar field. Thus, we restrict our study to a sperically symmetric spacetime (M, g) with a massless scalar field ϕ. In this case, the Klein-Gordon equation for ϕ is just the wave equation on (M, g) and the stress-energy-momentum tensor is:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi.$$

We obtain the Einstein scalar field equations:

$$\begin{cases} R_{\mu\nu} = 8\pi\partial_\mu\phi\partial_\nu\phi \\ \Box_g\phi = 0 \end{cases} \tag{1}$$

For this matter model, the Weak Cosmic Censorship conjecture can be formulated as follows:

Conjecture 3 (Weak Cosmic Censorship, Einstein Scalar Field in Spherical Symmetry). *For generic asymptotically flat initial data for the Einstein scalar field equations in spherical symmetry, the maximal globally hyperbolic future Cauchy development has a complete future null infinity* $\mathcal{I}^+$.

The Weak Cosmic Censorship conjecture was proved for the Einstein scalar field in spherical symmetry in Christodoulou's remarkable work [6], the first result of this kind. This built up on results previously obtained by Christodoulou in [3] – [5], while an overview of the entire proof of the conjecture is given in [7].

The current essay is an expository work based on [6], where we aim to present the proof of the instability of naked singularities for the Einstein scalar field equations in spherical symmetry. We also give a brief overview of the other elements that enter the proof of the Weak Cosmic Censorship. In section 2 we describe the general global structure of (smooth) spherically symmetric solutions to the Einstein scalar field equations, based on [3], [8], [9], and [10]. In section 3 we follow [4] to introduce a less regular class of solutions, namely the solutions of bounded variation, which is needed in the proof of the Weak Cosmic Censorship. In section 4 we present the criterion of [3] for the formation of trapped surfaces based on information on initial data. In section 5 we obtain that the existence of a trapped surface is a sufficient condition to have a complete $\mathcal{I}^+$, based on [8]. The center of the essay is section 6, where we present the proof in [6]. We introduce the blue-shift effect, and use it in section 6.1 to prove the main instability theorem. Finally, in section 6.2 we prove that the exceptional set of initial data leading to naked singularities has positive co-dimension in the set of integrable absolutely continuous initial data, completing the proof of the Weak Cosmic Censorship.

2 Global Structure of Spherically Symmetric Einstein Scalar Field Solutions

We consider spherically symmetric asymptotically flat initial data (Σ, h, k) for the Einstein scalar field equations, and denote by (M, g) its maximal globally hyperbolic future Cauchy development. Because of the spherical symmetry assumption, the quotient manifold $\mathcal{Q}^+ = J^+(\Sigma)/SO(3)$ inherits the structure of a $1+1$ Lorentzian manifold with boundary. We denote by Γ the portion of the boundary corresponding to the set of fixed points of the action of $SO(3)$, which we call the *center*. As a $1+1$ Lorentzian manifold, $\mathcal{Q}^+$ can be globally covered by double null coordinates (u, v), so the induced metric on $\mathcal{Q}^+$ is $-\Omega^2 dudv$. Then, the spacetime (M, g) can be globally[1] covered by double null coordinates (u, v, θ, ϕ). The metric on M is:

$$g = -\Omega^2 dudv + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where the function r represents the area radius of the group orbits.

Since the metric on $\mathcal{Q}^+$ is conformal to that of the $1+1$ dimensional Minkowski space, we can represent $\mathcal{Q}^+$ as a submanifold of $\mathbb{R}^{1+1}$ using Penrose diagrams. This approach offers a canonical way of attaching a boundary to the quotient $\mathcal{Q}^+$. We point out that we use the topology and causality of the ambient $\mathbb{R}^{1+1}$.

Under the assumption of spherical symmetry, the Einstein scalar field equations can be written as a system of PDEs on the quotient manifold $\mathcal{Q}^+$. In double null coordinates the Einstein equations in (1) become:

$$\partial_u(\Omega^{-2}\partial_u r) = -4\pi r\Omega^{-2}(\partial_u\phi)^2 \tag{2}$$

$$\partial_v(\Omega^{-2}\partial_v r) = -4\pi r\Omega^{-2}(\partial_v\phi)^2 \tag{3}$$

[1] Except of course for the usual coordinate singularities of the standard coordinates (θ, ϕ) on the sphere.

$$\partial_u\partial_v \log\Omega = -\frac{1}{r}\partial_u\partial_v r - 4\pi\partial_u\phi\partial_v\phi \tag{4}$$

$$\partial_u\partial_v r^2 = -\frac{1}{2}\Omega^2 \tag{5}$$

Equations (2) and (3) are the famous *Raychaudhuri equations*. We define the *Hawking mass*:

$$m = \frac{r}{2}(1 + 4\Omega^{-2}\partial_u r\partial_v r), \tag{6}$$

which satisfies the equations:

$$\partial_u m = -8\pi r^2\Omega^{-2}\partial_v r(\partial_u\phi)^2 \tag{7}$$

$$\partial_v m = -8\pi r^2\Omega^{-2}\partial_u r(\partial_v\phi)^2 \tag{8}$$

Finally, the wave equation in (1) can be written as:

$$r\partial_u\partial_v\phi + \partial_v\phi\partial_u r + \partial_u\phi\partial_v r = 0 \tag{9}$$

We consider the boundary conditions $r = 0,\ m = 0$ on Γ, and prescribe characteristic initial data (r, Ω, ϕ) on an outgoing null cone $C_0^+ := \{u = 0\}$, such that $r \in C^2,\ \Omega, \phi \in C^1$. We remark that the initial data must satisfy the constraint equation (2). To have asymptotically flat initial data, we also require that m is bounded on the initial hypersurface C_0^+, with supremum M_f. Under this conditions we obtain a solution $r \in C^2,\ \Omega, \phi \in C^1$ in $\mathcal{Q}^+$. In the view of the following sections, we shall refer to this as a "smooth" solution.

To obtain a smooth solution in $\mathcal{Q}^+$ as described above, we firstly need to have local well-posedness in the class of smooth initial data. This follows by a standard iteration argument. Then, in order for $\mathcal{Q}^+$ to be maximal, we also need an extension principle. In the following we use the notion of a first singularity as defined in [9]. We have the following extension principle from [10]:

Theorem 3. *Let $p \in \overline{\mathcal{Q}^+}\backslash\mathcal{Q}^+$ be a first singularity away from the center, i.e. $p \notin \overline{\Gamma}\backslash\Gamma$. For any $q \in \big(I^-(p) \cap \mathcal{Q}^+\big)\backslash\{p\}$ such that $\mathcal{D} = \big(J^+(q) \cap J^-(p)\big)\backslash\{p\} \subset \mathcal{Q}^+$, we have* $\inf_{\mathcal{D}} r = 0$.

In the following proposition we list some useful basic monotonicity properties satisfied by the solution:

Proposition 1. *For a smooth solution in $\mathcal{Q}^+$ we have that $\partial_u r < 0,\ m \geq 0,\ \partial_u\partial_v r \leq 0,\ \partial_v m \geq 0$. We also have that $\partial_v r$ and $1 - \frac{2m}{r}$ have the same sign, while $\partial_v r$ and $\partial_u m$ have opposite signs. Finally, in the causal past of Γ we have that $\partial_v r \geq 0$.*

Proof. Using the boundary condition $r = 0$ on Γ and the fact that $\partial_v r > 0$ at $\Gamma \cap C_0^+$, we get $\partial_u r < 0$ at $\Gamma \cap C_0^+$. By (5) we have that $\partial_u r < 0$ on C_0^+. We remark that the Raychaudhuri equations imply:

$$\partial_u(\Omega^{-2}\partial_u r) \leq 0,\ \partial_v(\Omega^{-2}\partial_v r) \leq 0.$$

In particular, $\partial_u r < 0$ in $\mathcal{Q}^+$. From equation (8) we get $\partial_v m \geq 0$, and using the boundary condition $m = 0$ on Γ we obtain that $m \geq 0$ in $\mathcal{Q}^+$. We use (5) to write $m = -2r^2\Omega^{-2}\partial_u\partial_v r$. Thus:

$$\partial_u\partial_v r \leq 0.$$

The rest of the proof follows in a similar fashion. $\square$

Definition 2. *We define the regular, trapped and marginally trapped regions as follows:*

$$\mathcal{R} = \{p \in \mathcal{Q}^+ : \ \partial_v r > 0\},$$

$$\mathcal{T} = \{p \in \mathcal{Q}^+ : \ \partial_v r < 0\},$$

$$\mathcal{A} = \{p \in \mathcal{Q}^+ : \ \partial_v r = 0\}.$$

We have the following result from [4], which proves that $\mathcal{T}$ is a future set and $\mathcal{A}$ is achronal:

Proposition 2. *Let $p = (u, v) \in \mathcal{Q}^+$. If $p \in \mathcal{T}$ then $J^+(p) \cap \mathcal{Q}^+ \subset \mathcal{T}$. If $p \in \mathcal{T} \cup \mathcal{A}$ then $J^+(p) \cap \mathcal{Q}^+ \subset \mathcal{T} \cup \mathcal{A}$ and for any $q = (u', v') \in J^+(p) \cap \mathcal{Q}^+$ with $u' > u$ we have $q \in \mathcal{T}$.*

We follow the discussion in [8] to introduce the concepts of future null infinity and event horizon. We define the set $\mathcal{U} = \{u : \ \sup_{(u,v)\in\mathcal{Q}^+} r(u, v) = \infty\}$. This set is nonempty because our data is asymptotically flat. We notice that for any $u \in \mathcal{U}$ there exists a unique $v^*(u)$ such that $\big(u, v^*(u)\big) \in \overline{\mathcal{Q}^+}\backslash\mathcal{Q}^+$.

Definition 3. *Future null infinity is the set:*

$$\mathcal{I}^+ = \bigcup_{u\in\mathcal{U}} \big(u, v^*(u)\big).$$

We interpret $\mathcal{I}^+$ as the set of far away observers. According to [8], we have that if $\mathcal{I}^+$ is nonempty, it is an ingoing null connected curve with $v = v_{\mathcal{I}^+}$. We define *timelike infinity* to be the point $i^+ = (\sup_{u\in\mathcal{U}} u, v_{\mathcal{I}^+})$. Using the monotonicity properties, one also gets that $J^-(\mathcal{I}^+) \cap \mathcal{Q}^+ \subset \mathcal{R}$.

Definition 4. *If $\mathcal{A} \cup \mathcal{T} \neq \emptyset$, we define the event horizon $\mathcal{H}^+$ to be the future boundary of $J^-(\mathcal{I}^+) \cap \mathcal{Q}^+$ in $\mathcal{Q}^+$. By the extension principle we get that $\mathcal{H} = \mathcal{Q}^+ \cap \{u = u_{\mathcal{H}}\}$, where $u_{\mathcal{H}} = \sup_{u\in\mathcal{U}} u$.*

According to [8] we also have that the event horizon is complete and the Penrose-type inequality $r \leq 2M_f$ holds. In particular this implies that $i^+ \notin \mathcal{I}^+$.

We now define the other possible components of the boundary as in [10]. Denote $b_\Gamma = \overline{\Gamma}\backslash\Gamma$. If $b_\Gamma = i^+$ we obtain that $\mathcal{Q}^+ = \mathcal{R}$ so the solution has the Penrose diagram of Minkowski space. We call such a solution *dispersive*. If $b_\Gamma \neq i^+$, we say that b_Γ is a first singularity at the center. We define the *central component of the singular boundary*:

$$\mathcal{B}_0 = \{(u_{b_\Gamma}, v) \in \overline{\mathcal{Q}^+}\}.$$

Finally, we define the *singular component of the boundary*:

$$\mathcal{B} = \{(u, v) \in \overline{\mathcal{Q}^+}\backslash\mathcal{Q}^+ : \ u_{i^+} < u < u_{b_\Gamma}\}.$$

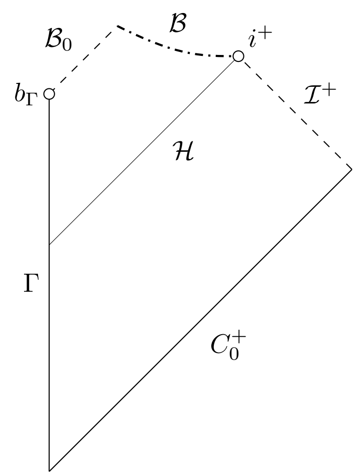
A priori we only have that $\mathcal{B}$ is a union of first singularities and null Cauchy horizons arising out of them. However, using the extension principle and the above two propositions it is straightforward to show that $\mathcal{B}$

is actually spacelike and $r = 0$ on $\mathcal{B}$. Similarly, we might a priori have an incoming null component of the boundary arising from i^+. But using the good monotonicity properties and the information that we have on $\mathcal{H}^+$ for this matter model, we see this is not possible.

According to the above discussion we have that the quotient spacetime obtained from a solution of the Einstein scalar field equations in spherical symmetry has the boundary:

$$\partial\mathcal{Q}^+ = C_0^+ \cup \mathcal{I}^+ \cup i^+ \cup \mathcal{B} \cup \mathcal{B}_0 \cup b_\Gamma \cup \Gamma.$$

We note that we might have $\mathcal{B} = \emptyset$ or $\mathcal{B}_0 = \{b_\Gamma\}$. In general the Penrose diagram of such spacetimes is:



We now prove a simple result that will be useful later:

Lemma 1. *Suppose that $b_\Gamma \neq i^+$, and consider the incoming null hypersurface $C_0^- := \{v = v_{b_\Gamma}\} \cap \mathcal{Q}^+$. Then $r \to 0$ as $u \to u_{b_\Gamma}$ on C_0^-.*

Proof. Using the fact that $r|_\Gamma = 0$ and $\partial_u\partial_v r \leq 0$ we get:

$$r(u, v_{b_\Gamma}) = \int_u^{v_{b_\Gamma}} \partial_v r(u,v)dv \leq \int_u^{v_{b_\Gamma}} \partial_v r(0,v)dv \to 0 \text{ as } u \to u_{b_\Gamma}.$$

□

3 Bounded Variation Solutions

An important principle when dealing with the initial value problem is that one should always work with a class of solutions for which we have well-posedness. In our situation, this should be the case both for solutions with naked singularities and for their perturbation to nearby solutions without naked singularities. So far, the discussion in the previous sections was in the class of smooth solutions, for which we have indeed well-posedness. While we expect the Weak Cosmic Censorship to hold for smooth solutions as well, we will allow for a rougher class of solutions, that of solutions of bounded variation. We will prove the Weak Cosmic Censorship for absolutely continuous initial data, which is contained in the class of bounded variation data. This approach is successful since the mechanism of instability of naked singularities is easier to excite in this rougher class of solutions. This section is based on Christodoulou's work in [4].

We prescribe characteristic initial data for the Einstein scalar field equations in spherical symmetry on a complete null outgoing hypersurface C_0^+. The double null coordinates are fixed by the conditions $u = 0$, $v = 2\arctan r$ on C_0^+, and $u = v$ on Γ.

Recall that the Einstein scalar field equations in spherical symmetry for the functions (r, Ω, ϕ) are the system of equations (2) – (5) and (9). We remark that the Gauss curvature of $\mathcal{Q}^+$, viewed as a surface in the spacetime $\mathcal{Q}^+ \times S^2$, is:

$$K_{\mathcal{Q}^+} = 4\Omega^{-2}\partial_u\partial_v \log\Omega.$$

A simple computation shows that equation (4) is implied by (2), (3), (5) and (9). We set aside equation (4) for now, and rewrite the remaining equations in a more convenient form.

Consider the system of equations for (r, Ω, ϕ):

$$\lambda = \partial_v r, \ \ \nu = \partial_u r \tag{10}$$

$$\theta = r\partial_v\phi, \ \ \zeta = r\partial_u\phi \tag{11}$$

$$\alpha = \partial_v(r\phi) = \theta + \phi\lambda, \ \ \beta = \partial_u(r\phi) = \zeta + \phi\nu \tag{12}$$

$$m = \frac{r}{2}(1 + 4\Omega^{-2}\partial_u r\partial_v r), \ \ \mu = \frac{2m}{r} \tag{13}$$

$$\partial_u\lambda = \frac{\mu}{1-\mu} \cdot \frac{\lambda\nu}{r} \tag{14}$$

$$\partial_v\nu = \frac{\mu}{1-\mu} \cdot \frac{\lambda\nu}{r} \tag{15}$$

$$2\nu\partial_u m = (1-\mu)\zeta^2 \tag{16}$$

$$2\lambda\partial_v m = (1-\mu)\theta^2 \tag{17}$$

$$\partial_u\theta + \frac{\lambda\zeta}{r} = 0 \tag{18}$$

$$\partial_v\zeta + \frac{\nu\theta}{r} = 0 \tag{19}$$

$$\partial_u\alpha = \phi\partial_u\lambda \tag{20}$$

$$\partial_v\beta = \phi\partial_v\nu \tag{21}$$

This system is equivalent to the Einstein scalar field equations in spherical symmetry. With the notation introduced by (10) – (13) we have that (14), (15) are equivalent to (5), equations (16), (17) are equivalent to Raychaudhuri's equations (2), (3), and equations (18) – (19) are equivalent to the wave equation (9). Using (12), we can rewrite equations (18) – (19) as (20) – (21).

We consider the boundary conditions $r|_\Gamma = 0$, $m|_\Gamma = 0$. Initial data to the characteristic initial value problem is given by (r, Ω, ϕ) on C_0^+. Recall that we already set $r = \tan \frac{1}{2}v$ on C_0^+. We prescribe the function ϕ on C_0^+, and by a translation $\phi \to \phi + c$ we can take $\phi(0,0) = 0$. By the regularity of the initial data we have that $(\lambda + \nu)(0,0) = 0$ and $\mu(0,0) = 0$, which gives $\Omega(0,0) = 1$. Using equation (2) on C_0^+, we obtain Ω on C_0^+. Thus, with our choice of coordinates we only need to prescribe ϕ on C_0^+.

Given a globally hyperbolic region $\mathcal{U} \subset \mathcal{Q}^+$, we introduce the following notation:

$$\mathcal{D}(u,v) = \left\{(\tilde{u}, \tilde{v}) \in \mathcal{U} : \ \tilde{u} \in [u,v), \tilde{v} \in (\tilde{u}, v]\right\}$$

$$a_* = \sup\left\{u : \ \exists v > u, \ (u,v) \in \mathcal{U}\right\}$$

Definition 5. *A triple (r, Ω, ϕ) defined in a globally hyperbolic region $\mathcal{U}$ is a solution of bounded variation to the Einstein scalar field equations in spherical symmetry if it satisfies equations (10) – (21) and:*

1. $\inf_{\mathcal{U}} \nu > -\infty$, $\inf_{\mathcal{D}(0,a)} \lambda > 0$ *for any $a \in (0, a_*)$*
2. *λ is of BV on each $C^+(u)$, uniformly in u and ν is of BV on each $C^-(v)$, uniformly in v*
3. *for all $a \in [0, a_*)$, we have* $\lim_{\epsilon \to 0+} (\lambda + \nu)(a, a + \epsilon) = 0$
4. *ϕ is AC on each $C^+(u)$, with TV bounded uniformly in u and ϕ is AC on each $C^-(v)$, with TV bounded uniformly in v*
5. *for all $a \in [0, a_*)$, we have:*

$$\lim_{\epsilon \to 0+} \sup_{0<\delta\leq\epsilon} TV_{\{a-\delta\}\times(a-\delta,a)}[\phi] = 0, \ \lim_{\epsilon \to 0+} \sup_{0\leq\delta<\epsilon} TV_{(a-\epsilon,a-\delta)\times\{a-\delta\}}[\phi] = 0$$

$$\lim_{\epsilon \to 0+} \sup_{0<\delta\leq\epsilon} TV_{(a,a+\delta)\times\{a+\delta\}}[\phi] = 0, \ \lim_{\epsilon \to 0+} \sup_{0\leq\delta<\epsilon} TV_{\{a+\delta\}\times(a+\delta,a+\epsilon)}[\phi] = 0$$

6. *θ is of BV on each $C^+(u)$, uniformly in u and ζ is of BV on each $C^-(v)$, uniformly in v*
7. *for all $a \in [0, a_*)$, we have* $\lim_{\epsilon \to 0+} (\alpha + \beta)(a, a + \epsilon) = 0$.

Remark 1. *In condition 1) in the above definition we only ask that λ is positive in the causal past of Γ in order to allow for BV solutions with trapped surfaces. Conditions 3), 5) and 7) are boundary conditions at the center which hold by default for more regular solutions (e.g. smooth solutions).*

Remark 2. *Given (f, g) functions on $\mathcal{U}$, we denote by $A(\mathcal{U})[f, g]$ the area of the image of $\mathcal{U}$ by the map $(u,v) \mapsto (f(u,v), g(u,v))$. If f, g are BV functions, the expression in terms of the Jacobian is interpreted as a Stieltjes integral. One obtains that for globally hyperbolic developments of BV data the areas $A(\mathcal{D}(0,a))[\lambda, \phi]$, $A(\mathcal{D}(0,a))[\nu, \phi]$, $A(\mathcal{D}(0,a))[\phi, \alpha/\lambda]$, and $A(\mathcal{D}(0,a))[\phi, \beta/\nu]$ are finite for any $a \in (0, a_*)$.*

Remark 3. *For a solution of bounded variation we have that the Gauss equation (4) holds in the sense of measures on $\mathcal{Q}^+$.*

The results in Proposition 1 and Proposition 2 hold for BV solutions as well. Moreover, one can prove that the following boundary conditions hold: $\lim_{v\to u+}\theta(u,v)=0$, $\lim_{u\to v-}\zeta(u,v)=0$ and $\lim_{v\to u+}\mu(u,v)=\lim_{u\to v-}\mu(u,v)=0$.

Define the function $\kappa=\frac{\lambda}{1-\mu}$, and note that in $\mathcal{R}\cup\mathcal{T}$ we have $\kappa>0$ and:

$$\partial_u\kappa=\frac{\theta^2}{\nu}\cdot\kappa \tag{22}$$

Thus, κ is globally defined and $\kappa>0$. It is useful to also write down the following equations, where (23) and (26) only make sense when $\lambda\neq 0$:

$$\partial_v\frac{\nu}{1-\mu}=\frac{\theta^2}{r\lambda}\cdot\frac{\nu}{1-\mu} \tag{23}$$

$$\partial_u\mu=\frac{1}{r}\left[(1-\mu)\frac{\zeta^2}{\nu}-\mu\nu\right] \tag{24}$$

$$\partial_v\mu=\frac{1}{r}\left[(1-\mu)\frac{\theta^2}{\lambda}-\mu\lambda\right] \tag{25}$$

$$\partial_u\frac{\alpha}{\lambda}=-\frac{\theta}{\lambda}\cdot\frac{\mu}{1-\mu}\cdot\frac{\nu}{r} \tag{26}$$

$$\partial_v\frac{\beta}{\nu}=-\frac{\zeta}{\nu}\cdot\frac{\mu}{1-\mu}\cdot\frac{\lambda}{r} \tag{27}$$

We want to establish well-posedness in the class of BV solutions. The following uniqueness result holds:

Theorem 4 (Uniqueness)**.** *Consider two BV solutions (r_1,Ω_1,ϕ_1) and (r_2,Ω_2,ϕ_2) defined in the globally hyperbolic regions $\mathcal{U}_1$, respectively $\mathcal{U}_2$. If the solutions have the same initial data on C_0^+, they coincide in the globally hyperbolic region $\mathcal{U}_1\cap\mathcal{U}_2$.*

According to the above discussion, to pose characteristic initial data we only need to prescribe ϕ on C_0^-. In the case of smooth solutions it is enough to prescribe α on C_0^-, which we can integrate to obtain ϕ. By a density argument it is sufficient to prescribe α on C_0^- for BV solutions as well. We have the following local existence result:

Theorem 5 (Local existence)**.** *For any initial data $\alpha_0\in BV\big((0,v_{\mathcal{I}^+})\big)$ there exists $\delta>0$ and a BV solution to the initial value problem defined in $\big\{(u,v):\ u\in[0,\delta),\ v\in(u,v_{\mathcal{I}^+})\big\}$.*

The proof is done using an approximation by smooth solutions. The key element is the *area estimates.* These replace the traditional energy estimates, and are adapted to the low regularity of the solution. One controls the areas $A[\lambda,\phi]$, $A[\nu,\phi]$, $A[\phi,\alpha/\lambda]$, and $.A[\phi,\beta/\nu]$ by the BV norm of the initial data, which allows a limiting argument. A very similar proof can be done to obtain local existence for the characteristic initial value problem with data on $\{u_0\}\times[v_0,v_1]\cup[u_0,u_1]\times\{v_0\}$. Thus, we have local well-posedness in the class of BV solutions. We now prove an extension principle similar to that in Theorem 3:

Theorem 6. *Consider a BV solution defined in the globally hyperbolic region $\mathcal{U}$. Let $p = (a, V) \in \overline{\mathcal{U}} \backslash \mathcal{U}$, such that $p \notin \overline{\Gamma}$ and $V < v_{\mathcal{I}^+}$. Suppose that there exists $q = (a_0, b) \in \big(I^-(p) \cap \mathcal{U}\big) \backslash \{p\}$ such that the region $\mathcal{D} = \big(J^+(q) \cap J^-(p)\big) \backslash \{p\}$ satisfies $\mathcal{D} \subset \mathcal{U}$, $\mathcal{D} \cap \Gamma = \emptyset$ and $\inf_{\mathcal{D}} r > 0$. Then the solution can be extended to have $(a, V) \in \mathcal{U}$.*

Proof. We begin by giving a brief outline of the strategy of the proof. The idea is to approximate the BV initial data by smooth initial data. For any $\epsilon > 0$ small enough, each smooth solution can be extended to a globally hyperbolic region where $r_n \geq \epsilon$. We control the pointwise and BV norms uniformly in terms of the pointwise and BV norms of the initial data. We then make a limiting argument to obtain a BV solution in a globally hyperbolic region with $r \geq \epsilon$. In particular, this will extend our initial BV solution to $\mathcal{D} \cup \{p\}$.

We notice that $\mathcal{D} = [a_0, a] \times [b, V] \backslash \{(a, V)\}$. Since $(a_0, V) \in \mathcal{U}$, there exists $V_1 \in (V, v_{\mathcal{I}^+})$ such that $\{a_0\} \times [b, V_1] \subset \mathcal{U}$. We approximate the BV initial data induced on $\{a_0\} \times [b, V_1] \cup [a_0, a] \times \{b\}$ by smooth initial data which is uniformly bounded in the C^0 and BV norms. Consider $\epsilon > 0$ small enough, such that $\inf_{\mathcal{D}} r > 2\epsilon$ and $r_n > 2\epsilon$ on $\{a_0\} \times [b, V_1] \cup [a_0, a] \times \{b\}$ for all n. We use local well-posedness and the extension principle for smooth solutions from section 2. We get that for every n there exists $v_{\epsilon,n}(a) \in (b, V_1]$ with the property that the smooth solution (r_n, Ω_n, ϕ_n) can be extended to $[a_0, a] \times [b, v_{\epsilon,n}(a))$ such that $r_n \geq \epsilon$ in this region, and if $v_{\epsilon,n}(a) < V_1$ then $\lim_{v \to v_{\epsilon,n}(a)-} r_n(a, v) = \epsilon$.

We now prove that each smooth solution is uniformly bounded in the pointwise and BV norms on $[a_0, a] \times [b, v_{\epsilon,n}(a))$ in terms of the pointwise and BV norms on $\{a_0\} \times [b, v_{\epsilon,n}(a)) \cup [a_0, a] \times \{b\}$. The pointwise bounds are proved in [4]. Then, the pointwise bounds for θ_n, ζ_n imply that $TV[\phi_n(u, \cdot)]$, $TV[\phi_n(\cdot, v)]$ are uniformly bounded. We use the equations for $\partial_u \nu_n$ and $\partial_u \beta_n$ obtained by integrating the equations for $\partial_v \partial_u \nu_n$ and $\partial_v \partial_u \beta_n$, and we get $TV[\nu_n(\cdot, v)]$, $TV[\beta_n(\cdot, v)]$ are uniformly bounded. Then, we use the equation:

$$\partial_u \partial_v \kappa_n = \partial_v \kappa_n \cdot \frac{\zeta_n^2}{r_n \nu_n} + \kappa_n \partial_v \frac{\zeta_n^2}{r_n \nu_n}$$

and Gronwall's lemma to obtain that $|\partial_v \kappa_n|$ is uniformly bounded, which gives $TV[\lambda_n(u, \cdot)]$ is uniformly bounded. In particular, this implies $A[\lambda_n, \phi_n]$ is bounded. Finally, by the area estimate:

$$TV[\alpha_n(u, \cdot)] \leq TV[\alpha_n(0, \cdot)] + \sup |\phi_n| TV[\lambda_n(u, \cdot)] + A[\lambda_n, \phi_n],$$

we conclude that $TV[\alpha_n(u, \cdot)]$ is also bounded.

So far we obtained a sequence of smooth solutions (r_n, Ω_n, ϕ_n) defined on $[a_0, a] \times [b, v_{\epsilon,n}(a))$, which is pointwise and BV uniformly bounded. We define $v_\epsilon(a) = \limsup v_{\epsilon,n}(a)$, and we remark that $v_\epsilon(a) > b$ because $|\lambda_n|$ is uniformly bounded and $r_n > 2\epsilon$ on $[a_0, a] \times \{b\}$. Similarly to the proof of Theorem 5, one can show that a subsequence of (r_n, Ω_n, ϕ_n) converges to a BV solution (r, Ω, ϕ) defined on $[a_0, a] \times [b, v_\epsilon(a))$. Moreover, since $r_n \geq \epsilon$ we have that $r \geq \epsilon$. If $v_\epsilon(a) < V_1$, we have $v_{\epsilon,n}(a) < V_1$ for n large enough, so $\lim_{v \to v_{\epsilon,n}(a)-} r_n(a, v) = \epsilon$. Since $|\lambda_n|$ is uniformly bounded, we also get $\lim_{v \to v_\epsilon(a)-} r(a, v) = \epsilon$. Thus, $\inf_{\mathcal{D}} r > 2\epsilon$ implies that $v_\epsilon(a) > V$. In the view of the uniqueness theorem for BV solutions, we extended our original BV solution to $[a_0, a] \times [b, v_\epsilon(a)) \supset \mathcal{D} \cup \{p\}$. □

We state an extension principle at the center, which is essential in the proof of the Weak Cosmic Censorship:

Theorem 7. *Given initial data $\alpha_0 \in AC(0,V)$, we consider the solution (r,Ω,ϕ) to the initial value problem. Let $v_0 < V$ be such that for all $v < v_0$ we have $\mathcal{D}(0,v) \subset \mathcal{Q}^+$. If $\mu(u,v_0) \to 0$ as $u \to v_0-$, there exists $\delta > 0$ such that $\mathcal{D}(0,v_0+\delta) \subset \mathcal{Q}^+$.*

Remark 4. *Unfortunately, this result does not apply if the initial data is only assumed to be BV. This can be seen in the example from [4] of a scale invariant solution with $k^2 > 1$, which is flat in the causal past of C_0^-, but has a spacelike singular boundary $\mathcal{B}$ arising from b_Γ.*

Remark 5. *We mention an interesting continuity property of BV solutions. If $\alpha(u_0,\cdot)$ is continuous at v, then $\alpha(u,\cdot)$ is continuous at v for all u such that $(u,v) \in \mathcal{U}$. Thus, α is either continuous across the hypersurface $C^-(v)$, or it jumps across it along all outgoing null cones. Similar results hold for λ, θ, β, ν and ζ. We also obtain that for any $v \in (0, v_{\mathcal{I}^+})$ such that $\lambda(0,\cdot), \alpha(0,\cdot)$ and $\theta(0,\cdot)$ are continuous at v, all the equations with a ∂_v derivative in (10) – (21) hold pointwise on $C^-(v)$, and similarly on $C^+(u)$.*

4 Formation of Trapped Surfaces

We mentioned in the introduction that the Incompleteness Theorem of Penrose assumes the existence of a trapped surface in the initial data. However, we are interested in collapse of regular initial data. Thus, it is essential to have a criterion for the formation of trapped surfaces based on information on initial data. The following is the main result of [3]:

Theorem 8. *On the outgoing null cone C_0^+ we consider two spheres $S_{1,0}$ and $S_{2,0}$, with $S_{2,0}$ to the exterior of $S_{1,0}$. We define:*

$$\delta_0 := \frac{r_{2,0}}{r_{1,0}} - 1, \;\; \eta_0 = \frac{2(m_{2,0} - m_{1,0})}{r_{2,0}}.$$

We denote the incoming future null cones through $S_{1,0}$ and $S_{2,0}$ by C_1^- and C_2^-. Assume that $r \to 0$ along C_1^-. There exist constants $c_1, c_2 > 0$ such that if:

$$\delta_0 \leq c_0, \;\; \eta_0 > c_1 \delta_0 \log\left(\frac{1}{\delta_0}\right),$$

then there exists an outgoing null cone C_*^+ such that the sphere $S_{2*} = C_*^+ \cap C_2^-$ is marginally trapped.

Intuitively, this criterion states that if the mass within an annular region bounded by two spheres on C_0^+ is large compared to the dimensions of the region, then a trapped surface will form in evolution. We refer the reader to [3] for a proof of the theorem.

Remark 6. *As stated in [3], the theorem assumes the existence of a sphere* $S_2' = C_+' \cap C_2^-$ *such that:*

$$r_2' = \frac{3\delta_0}{1+\delta_0} r_{2,0},$$

which can be arbitrarily small. However, when applying this theorem we do not want to assume smallness of r *on* C_2^-*. This can be avoided by noticing that* $r \to 0$ *along* C_1^- *implies the existence of a sphere* $S_1' = C_+' \cap C_1^-$ *such that* $r_1' = 2\delta_0 r_{1,0}$*. Now, the condition* $\partial_u \partial_v r \leq 0$ *and the extension principle give a sphere* S_2' *as desired.*

Remark 7. *In addition to using the theorem as a criterion for the formation of trapped surfaces, we will also use it as part of the contradiction argument when proving the Weak Cosmic Censorship. Thus, if we assume that the spacetime has no marginally trapped region, the contrapositive of the theorem gives us an estimate for the mass contained within a small annular region to the exterior of* C_1^-*.*

5 Existence of Trapped Surfaces implies Future Null Infinity is Complete

We recall that the statement of Weak Cosmic Censorship is that, generically, far away observers do not see singularities, i.e. future null infinity $\mathcal{I}^+$ is complete. A general definition of this is given in [7] for maximal developments of asymptotically flat initial data, without reference to the conformal compactification of the spacetime (and without assuming spherical symmetry). Following [8], we translate this definition to the spherically symmetric case into a simpler condition.

For the rest of the section we assume that the spacetime contains a (marginally) trapped surface, so $\mathcal{A} \cup \mathcal{T} \neq \emptyset$. We define the event horizon as in section 2, and we write $\mathcal{H} = \mathcal{Q}^+ \cap \{u = u_{\mathcal{H}}\}$.

Fix an outgoing null ray $\{u = u_0\}$ with $u_0 < u_{\mathcal{H}}$. The geodesic equation for incoming null geodesics is:

$$\ddot{u} + \partial_u \log \Omega^2 (\dot{u})^2 = 0.$$

We integrate this to obtain the vector field:

$$\frac{\Omega^2(u_0,v)}{\Omega^2(u,v)} \cdot \frac{\partial}{\partial u} = \frac{\partial_v r(u_0,v)}{\partial_v r(u,v)} \cdot \frac{1-\mu(u,v)}{1-\mu(u_0,v)} \cdot \frac{\partial_u r(u_0,v)}{\partial_u r(u,v)} \cdot \frac{\partial}{\partial u}$$

which is tangent to all incoming null geodesics, and parallel along the outgoing null ray $\{u = u_0\}$. The affine length of these incoming null geodesics is:

$$l(v) = \int_{u_0}^{u_{\mathcal{H}}} \frac{\partial_v r(u,v)}{\partial_v r(u_0,v)} \cdot \frac{1-\mu(u_0,v)}{1-\mu(u,v)} \cdot \frac{\partial_u r(u,v)}{\partial_u r(u_0,v)} du,$$

where the fact that the affine length is zero on $\{u = u_0\}$ and that the vector field is parallel along the ray $\{u = u_0\}$ are the necessary normalization conditions.

We say that the spherically symmetric maximal future Cauchy development $(\mathcal{M}, g)$ has a *complete future null infinity* if in the quotient spacetime $\mathcal{Q}^+$ we have $l(v) \to \infty$ as $v \to v_{\mathcal{I}^+}$. The main result of [8] is the following:

Theorem 9. *If $\mathcal{A} \neq \emptyset$, then $\mathcal{I}^+$ is future complete.*

We refer the reader to the original paper for the proof. We point out that a key result in the proof is the Penrose-type inequality $r \leq 2M_f$ on $\mathcal{H}$, which we mentioned in section 2. We also remark that the proof in assumes that $r \in C^2(\mathcal{Q}^+),\ \Omega, \phi \in C^1(\mathcal{Q}^+)$, using the extension principle proved in [9]. In order to have that the same proof works for a BV solution, we use the extension principle in Theorem 6.

Following the discussion in this section, we see that in order to prove the Weak Cosmic Censorship it suffices to show:

Conjecture 4. *For generic asymptotically flat initial data for the Einstein scalar field equations in spherical symmetry, the quotient spacetime $\mathcal{Q}^+$ of the maximal globally hyperbolic future Cauchy development satisfies:*

- *$\mathcal{Q}^+$ disperses, i.e. $b_\Gamma = i^+$, or*
- *$\mathcal{A} \neq \emptyset$.*

6 Instability of Naked Singularities

In this section we present the proof of the Weak Cosmic Censorship, following [6]. The main idea is that spacetimes that do not disperse have an instability mechanism at the center along C_0^-, namely the blue-shift effect. This forces solutions with naked singularities to be finely tuned, and in particular unstable.

We consider initial data $\alpha_0 \in AC\big([0, v_{\mathcal{I}^+})\big) \cap L^1\big([0, v_{\mathcal{I}^+})\big)$. If $b_\Gamma = i^+$ then $\mathcal{Q}^+ = \mathcal{R}$, and the fact that C_0^+ is complete implies that $\mathcal{I}^+$ is future complete. According to the section 5, if $\mathcal{A} \neq \emptyset$, we again have that $\mathcal{I}^+$ is future complete. For the rest of the section we assume that $b_\Gamma \neq i^+$ and $\mathcal{A} = \emptyset$.

We remark that so far the double null coordinates (u, v) were fixed by taking $v = 2\arctan r$ on $u = 0$ and $u = v$ on Γ. By Lemma 1, we know that $r \to 0$ as $u \to u_{b_\Gamma}-$ along C_0^-, so we can change the coordinate u to have $u = -2r$ on C_0^-. Let a be the radius of the sphere $C_0^- \cap C_0^+$, and set $a_0 = 2\arctan a$. We translate the coordinate v by a_0 in order to have $C_0^- = \{v = 0\}$ and $C_0^+ = \{u = -2a\}$. These choices fix the double null coordinates (u, v), but we now do not have $\Gamma = \{u = v\}$ anymore. However, the new coordinates are adapted for our problem.

Remark 8. *We claim that BV solutions have the same properties in the new coordinates. For the purpose of this remark, denote the new coordinates by $\tilde{u} = -2r(u, a_0),\ \tilde{v} = v - a_0$. The only functions that change are $\tilde{\nu}(u, v) = -\nu(u, v)/2\nu(u, a_0)$, $\tilde{\zeta}(u, v) = -\zeta(u, v)/2\nu(u, a_0)$, $\tilde{\beta}(u, v) = -\beta(u, v)/2\nu(u, a_0)$, and $\tilde{\Omega}^2(u, v) = -\Omega^2(u, v)/2\nu(u, a_0)$. Since $\inf_{\mathcal{U}} \nu > -\infty$ and $\sup_{\mathcal{U}} \nu < 0$ we have that $(\tilde{u}, \tilde{v})$ define global coordinates on $\mathcal{U}$ and conditions $1), 2), 4), 5)$ and $6)$ in the definition of a BV solution still hold. Moreover, we still have the monotonicity properties in Proposition 1, 2 and the boundary conditions $\theta = 0,\ \mu = 0$ at $\Gamma \cap C^+$ and $\zeta = 0,\ \mu = 0$ at $\Gamma \cap C^-$. Thus, we only need to adjust conditions $2)$ and $7)$. The new boundary conditions are obtained by formally taking derivatives of $r|_\Gamma = 0$ and $(r\phi)|_\Gamma = 0$ along $\Gamma = \{u(\tilde{u}, \tilde{v}) = v(\tilde{u}, \tilde{v})\}$.*

Using equations (14) and (18) we obtain along C_0^- :

$$\partial_u\left(\frac{\theta}{\lambda}\right) = -\frac{1}{u}\cdot\frac{\mu}{1-\mu}\cdot\frac{\theta}{\lambda} - \frac{1}{u}\cdot\frac{\zeta}{\nu}. \tag{28}$$

We define the integrating factor:

$$\gamma(u) = -\int_{-2a}^{u}\frac{1}{u'}\cdot\frac{\mu}{1-\mu}(u',0)du', \tag{29}$$

which is a physical measure of the blue-shift. The next two results support this interpretation:

Lemma 2. *We denote $\varkappa = \frac{1}{1-\mu}$. Then for all $u \in [-2a, 0)$:*

$$\varkappa(u,0) \leq 2\varkappa(-2a,0)e^{\gamma(u)}$$

Proof. Since m is non-increasing along C_0^- we get:

$$\mu(u,0) \leq \mu(-2a,0)\cdot\frac{a}{r(u,0)} \leq \left(1 - \frac{1}{\varkappa(-2a,0)}\right)\cdot\frac{a}{r(u,0)}$$

Thus, if $r(u,0) \geq a\cdot\big(1-\frac{1}{\varkappa(-2a,0)}\big)\cdot\big(1-\frac{1}{2\varkappa(-2a,0)}\big)^{-1}$, we have that $\varkappa(u,0) \leq 2\varkappa(-2a,0)$. Also for any $u' \in [-2a, u]$ we have:

$$\varkappa(u',0) - 1 = \frac{\mu(u',0)}{1-\mu(u',0)} \geq \frac{\mu(u,0)\cdot\frac{u}{u'}}{1-\mu(u,0)\cdot\frac{u}{u'}}.$$

This implies:

$$\gamma(u) \geq -\int_{-2a}^{u}\frac{1}{u'}\cdot\frac{\mu(u,0)\cdot\frac{u}{u'}}{1-\mu(u,0)\cdot\frac{u}{u'}}du' = \log\left(\frac{1+\frac{u}{2a}\mu}{1-\mu}\right) > \log\left(\frac{1-\frac{r}{a}}{1-\mu(u,0)}\right).$$

We conclude that if $r(u,0) < a\cdot\big(1-\frac{1}{\varkappa(-2a,0)}\big)\cdot\big(1-\frac{1}{2\varkappa(-2a,0)}\big)^{-1}$, we have that

$$\varkappa(u,0) < \frac{e^{\gamma(u)}}{1-\frac{r}{a}} < \big(2\varkappa(-2a,0)-1\big)e^{\gamma(u)}.$$

□

Proposition 3. *If γ is bounded, then $\mu(u,0)\to 0$ as $u \to 0-$.*

Proof. As in the proof of the previous lemma, we have for any $u_1 < u_2 \in [-2a, 0)$:

$$\gamma(u_2) - \gamma(u_1) \geq -\int_{u_1}^{u_2}\frac{1}{u'}\cdot\frac{\mu(u_2,0)\cdot\frac{u_2}{u'}}{1-\mu(u_2,0)\cdot\frac{u_2}{u'}}du' = \log\left(\frac{1-\frac{u_2}{u_1}\mu(u_2,0)}{1-\mu(u_2,0)}\right).$$

This implies:

$$\mu(u_2,0) \leq \frac{e^{\gamma(u_2)-\gamma(u_1)}-1}{1-\frac{u_2}{u_1}}.$$

Taking $u_1 = 2u_2$, we see that γ bounded implies that $\gamma(u_2)-\gamma(u_1)\to 0$ as $u_2 \to 0-$. □

Corollary 1. *Consider a spacetime $\mathcal{Q}^+$ which is the maximal globally hyperbolic future development of initial data $\alpha_0 \in AC\big([-a_0, v_{\mathcal{I}^+})\big) \cap L^1\big([-a_0, v_{\mathcal{I}^+})\big)$. If $b_\Gamma \neq i^+$, we have that the blue-shift γ is unbounded.*

Proof. We use the extension principle in Theorem 7 and the fact that $\mathcal{Q}^+$ cannot be extended at the center past C_0^-. □

Since $r \to 0$ along C_0^- and $\mathcal{A} = \emptyset$, we can apply the contrapositive of Theorem 8 to get a necessary condition to not form trapped surfaces. For any $\epsilon > 0,\ c > 0$ we consider the regions:

$$\mathcal{R}_\epsilon = \big\{(u,v) \in \mathcal{Q}^+ :\ u \in [-2a, 0),\ v \in [0,\epsilon]\big\},$$

$$\mathcal{R}_\epsilon^c = \left\{(u,v) \in \mathcal{Q}^+ :\ u \in [-2a, 0),\ v \in [0,\epsilon],\ \frac{r(u,v)}{r(u,0)} \leq e^c\right\}.$$

We also define the function[2]:

$$s(u,v) = \log\left(\frac{r(u,v)}{r(u,0)}\right),$$

and we note that $s = 0$ on C_0^-, and $s > 0$ for $v > 0$. With this notation we have for any $(u,v) \in \mathcal{R}_\epsilon$:

$$\delta(u,v) = \frac{r(u,v)}{r(u,0)} - 1 = e^s - 1 \tag{30}$$

$$\eta(u,v) = \frac{2\big(m(u,v) - m(u,0)\big)}{r(u,v)} = \mu(u,v) - \mu(u,0) \cdot e^{-s} \tag{31}$$

By Theorem 8, we get that there exist constants c_0, c_1 such that for any $(u,v) \in \mathcal{R}_\epsilon$, if $\delta(u,v) \leq c_0$ then:

$$\eta(u,v) < c_1 \delta(u,v) \log \frac{1}{\delta(u,v)},$$

or otherwise a trapped surface would form. Thus, we get that there exist (different) constants $c_0, c_1 > 0$ such that:

$$\eta(u,v) < c_1 s(u,v) \log \frac{1}{s(u,v)}\ \ \forall\ (u,v) \in \mathcal{R}_\epsilon^{c_0} \tag{32}$$

To summarize, we see that imposing the condition that $\mathcal{I}^+$ is future incomplete in $\mathcal{Q}^+$, we obtain the following restrictions:

- the blue-shift γ is unbounded along the incoming null hypersurface C_0^-,
- condition (32) holds in a small region $\mathcal{R}_\epsilon^{c_0}$ in the exterior of C_0^-.

Remark 9. *We provide a heuristic argument for why one should expect to have the additional estimate in (32). Firstly, consider the linear wave equation on a fixed background spacetime with a naked singularity. We claim that using the infinite blue-shift on C_0^-, one obtains that the energy of the solution on $\mathcal{R}_\epsilon \cap C(u)^+$ blows up as $u \to 0-$. We now imagine that we start with a spacetime with a naked singularity and we "add"*

[2] In [6] s is a coordinate, but we prefer to do everything in double null coordinates (u, v).

the nonlinear effects from equations (14) − (19). *We want to apply the idea from the linear case to the wave equation (18). However, the nonlinear effects may alter the metric so much that the solution may not be defined in the diamond* $[-2a, 0) \times [0, \epsilon]$ *anymore. In particular, a trapped surface could form arbitrarily close to* b_Γ. *Thus, in order to use the blue-shift effect as we did in the linear case, we need an extra smallness assumption that ensures the metric is close to that of a naked singularity spacetime. Therefore, we interpret condition (32) as the assumption needed so that we can use the "linear blue-shift effect" to study (18).*

6.1 An Instability Theorem

In this section we shall make use of the blue-shift to prove that unless the solution satisfies a restrictive property, a trapped surface forms arbitrarily close to C_0^-, contradicting (32). In this sense we say that the blue-shift is the instability mechanism for the Weak Cosmic Censorship problem.

To integrate equation (28), we also define the integral:

$$I(u) = \int_{-2a}^{u} \frac{1}{u'} \cdot \frac{\zeta}{\nu}(u', 0) \cdot e^{-\gamma(u')} du'. \tag{33}$$

We obtain along C_0^- :

$$\frac{\theta}{\lambda}(u, 0) = e^{\gamma(u)} \left(\frac{\theta}{\lambda}(-2a, 0) - I(u) \right). \tag{34}$$

We claim that this equation gives a very restrictive condition that our spacetime must satisfy in order to allow for naked singularities, which is $\lim_{u \to 0-} I(u) = \frac{\theta}{\lambda}(-2a, 0)$. Indeed, if we assume the contrary, then the fact that γ is unbounded implies that $\frac{\theta}{\lambda}(u, 0)$ is unbounded. It is then reasonable to expect that $\frac{\theta}{\lambda}(u, \cdot)$ is large on $v \in [0, \epsilon)$, for some u large enough. However, this will contradict (32), because by (16) we have:

$$\eta(u, v) = \frac{1}{r(u, v)} \int_0^v \frac{1}{\varkappa} \cdot \frac{\theta^2}{\lambda}(u, v') dv' \tag{35}$$

We make this idea precise in the following instability result. This theorem is the key result in [6]:

Theorem 10. *Suppose that* $b_\Gamma \neq i^+$ *and* $\mathcal{A} = \emptyset$. *Then:*

$$\lim_{u \longrightarrow 0-} I(u) = \frac{\theta}{\lambda}(-2a, 0).$$

Proof. We suppose that either I does not tend to a finite limit as $u \to 0-$, or that:

$$\lim_{u \longrightarrow 0-} I(u) \neq \frac{\theta}{\lambda}(-2a, 0) \tag{36}$$

We notice that we have no a priori information on the integral $I(u)$. Thus, considering the limits:

$$l_+ = \limsup_{u \longrightarrow 0-} I(u) \text{ and } l_- = \liminf_{u \longrightarrow 0-} I(u),$$

we have $-\infty \leq l_- \leq l_+ \leq \infty$, and one needs to consider all the possible cases. We restrict to the situation where $I(u)$ is bounded, because the cases where $|l_+| = \infty$ or $|l_-| = \infty$ are similar. We have that $\frac{\theta}{\lambda}(u,0)e^{-\gamma(u)}$ is also bounded, so we set:

$$b = \sup_{u\in[-2a,0)} \left|\frac{\theta}{\lambda}(u,0)e^{-\gamma(u)}\right|.$$

If $-\infty < l_- = l_+ = l < \infty$, we have that $\lim_{u\to 0-} I(u) = l$ and $l \neq \frac{\theta}{\lambda}(-2a,0)$. We set $h = \left|\frac{\theta}{\lambda}(-2a,0) - l\right|$. Then there exists $U \in [-2a,0)$ such that $|I(u) - l| \leq \frac{2h}{3}$ for all $u \in [U,0)$. This implies:

$$\left|\frac{\theta}{\lambda}(u,0)\right| = e^{\gamma(u)}\left|\frac{\theta}{\lambda}(-2a,0) - I(u)\right| \geq \frac{h}{3}e^{\gamma(u)} \ \forall u \in [U,0).$$

If $-\infty < l_- < l_+ < \infty$, we set $h = l_+ - l_-$. We get that:

$$\max\left\{\left|\frac{\theta}{\lambda}(-2a,0) - l_-\right|, \left|\frac{\theta}{\lambda}(-2a,0) - l_+\right|\right\} \geq \frac{h}{2},$$

so there exists a sequence $u_n \to 0-$ such that $\left|\frac{\theta}{\lambda}(-2a,0) - I(u_n)\right| \geq \frac{h}{3}$, which implies:

$$\left|\frac{\theta}{\lambda}(u_n,0)\right| = e^{\gamma(u_n)}\left|\frac{\theta}{\lambda}(-2a,0) - I(u_n)\right| \geq \frac{h}{3}e^{\gamma(u_n)}.$$

Therefore, in the cases when $I(u)$ is bounded, we have a sequence $u_n \to 0-$ such that:

$$\left|\frac{\theta}{\lambda}(u_n,0)\right| \geq \frac{h}{3}e^{\gamma(u_n)} \tag{37}$$

Since we know that $\frac{\theta}{\lambda}$ is unbounded on C_0^-, according to our outlined plan we next want to show that it is large enough in a small neighborhood of C_0^-. To measure this we consider the function:

$$\psi(u,v) = e^{-\gamma(u)}\left(\frac{\theta}{\lambda}(u,v)\cdot\frac{r(u,v)}{r(u,0)} - \frac{\theta}{\lambda}(u,0)\right) \tag{38}$$

We also define:

$$\omega(u,v) = \frac{u\nu}{r}(u,v)\big(\varkappa(u,v) - 2\big) - \big(\varkappa(u,0) - 2\big),$$
$$\rho(u,v) = e^{-\gamma(u)}\left(\omega(u,v)\cdot\frac{\theta}{\lambda}(u,0) + \xi(u,v)\right),$$

where we have that:

$$\xi(u,v) = \int_0^v \frac{\nu\theta}{r}(u,v')dv'$$

satisfies $-\zeta(u,v) = -\zeta(u,0) + \xi(u,v)/2$ by equation (19). With these definitions, the wave equation (18) along C_0^- and $C^-(v)$ implies:

$$-u\partial_u\psi = \omega\psi + \rho \tag{39}$$

We want to use (37) and (38) to get that $\frac{\theta}{\lambda}$ is bounded from below. Thus, we need to bound ψ. According to (39) we firstly need estimates on ω, ξ, ρ.

Step 1. Estimates for ω, ξ, and ρ.

We remark that in the process of proving estimates we also need to determine the small region in the exterior of C_0^- where the estimates hold. We denote the conditions:

$$(u,v) \in \mathcal{R}_\epsilon^{c_0} \tag{C1}$$

$$s(u,v)\log\frac{1}{s(u,v)} \leq \frac{e^{-\gamma(u)}}{4c_1\varkappa(-2a,0)} \tag{C2}$$

Thus, if $(C1)-(C2)$ hold for (u,v) we have:

$$\varkappa(u,v) = \frac{1}{1-\mu(u,v))} = \frac{1}{1-\mu(u,0)e^{-s}-\eta(u,v)} \leq \frac{1}{\frac{1}{\varkappa(u,0)}-\eta(u,v)} \leq \frac{1}{\frac{e^{-\gamma}}{2\varkappa(-2a,0)}-c_1 s\log\frac{1}{s}},$$

where we used Lemma 2 and (32). Then, by (C2) we obtain that if $(C1)-(C2)$ hold for (u,v):

$$\varkappa(u,v) \leq 4\varkappa(-2a,0)e^{\gamma(u)}. \tag{40}$$

Note that since $c_0 < 1/e$ and $c_1 > 1$ we get that $4\varkappa(-2a,0)e^{\gamma(u)}s \leq 1/c_1$ and $\mu(u,v) \leq c_1\log(1/s)$. Also:

$$\varkappa(u,v)-\varkappa(u,0) = \varkappa(u,v)\varkappa(u,0)\big(\mu(u,v)-\mu(u,0)\big) = \varkappa(u,v)\varkappa(u,0)\big[\eta(u,v)-\mu(u,0)(1-e^{-s(u,v)})\big]$$

We get that if $(C1)-(C2)$ hold, then:

$$|\varkappa(u,v)-\varkappa(u,0)| \leq 8\varkappa(-2a,0)^2e^{2\gamma(u)}s\log\frac{1}{s}.$$

Using equation (15) and the fact that $\nu = -\frac{1}{2}$ on C_0^- we get:

$$\log(-2\nu) \leq \big(4\varkappa(-2a,0)e^{\gamma(u)}-1\big)\log\frac{r(u,v)}{r(u,0)},$$

which implies that:

$$\frac{u\nu}{r}(u,v) \leq \exp\big(4\varkappa(-2a,0)e^{\gamma(u)}s(u,v)\big) \leq e^{1/c_1}. \tag{41}$$

Using the inequality $x-1 \leq \frac{c-1}{\log c}\log x$ whenever $x \in [1,c]$, the above also gives:

$$\frac{u\nu}{r}(u,v)-1 \leq 4c_1(e^{1/c_1}-1)\varkappa(-2a,0)e^{\gamma(u)}s(u,v).$$

But since $\lambda > 0$ we also get from (15) that:

$$1-\frac{u\nu}{r}(u,v) \leq 1+\frac{u}{2r}(u,v) = 1-e^{-s} \leq s.$$

Thus, we proved:

$$\left|1-\frac{u\nu}{r}(u,v)\right| \leq 4c_1e^{1/c_1}\varkappa(-2a,0)e^{\gamma(u)}s(u,v) \tag{42}$$

We can write:

$$\omega(u,v) = \frac{u\nu}{r}(u,v)\big(\varkappa(u,v)-\varkappa(u,0)\big)+\left(1-\frac{u\nu}{r}(u,v)\right)\cdot\big(2-\varkappa(u,0)\big)$$

We get that if $(C1)-(C2)$ hold, then for $c_2 = 16c_1^2 e^{1/c_1}$:

$$|\omega(u,v)| \le c_2 \varkappa(-2a,0)^2 e^{2\gamma(u)} s(u,v) \log \frac{1}{s(u,v)} \tag{43}$$

By Cauchy-Schwarz and (17), we have that:

$$\xi^2 \le \int_0^v \frac{\lambda}{r^2}(u,v')dv' \int_0^v 2\partial_v m \cdot \frac{\nu^2}{1-\mu}(u,v')dv' \le (e^s - 1)\eta \frac{\nu^2}{1-\mu}(u,v),$$

where we also used (15) and (23). It is straightforward to see that if $(C1)-(C2)$ hold, then using (32), (40) and (41) we get:

$$|\xi(u,v)| \le c_2 \varkappa(-2a,0)^2 e^{2\gamma(u)} s(u,v) \log \frac{1}{s(u,v)} \tag{44}$$

Using (43) and (44) in the definition of ρ, we get:

$$|\rho(u,v)| \le c_2 \varkappa(-2a,0)^2 (b+1) e^{2\gamma(u)} s(u,v) \log \frac{1}{s(u,v)} \tag{45}$$

Step 2. Estimates for ψ.

Recall that in order to show that $\frac{\theta}{\lambda}$ is bounded from below, we need estimates for ψ. So far we know that ψ satisfies the transport equation (39), and we have the estimates (43), (45) on the other terms in the equation. Thus, if $(C1)-(C2)$ hold, then:

$$\partial_u |\psi| \le \frac{1}{-u} c_2 \varkappa(-2a,0)^2 e^{2\gamma(u)} s(u,v) \log \frac{1}{s(u,v)} \big(|\psi| + b + 1\big)$$

We consider the integrating factor:

$$\varphi(u,v) = c_2 \varkappa(-2a,0)^2 \int_{-2a}^{u} \frac{1}{-u'} e^{2\gamma(u')} s(u',v) \log \frac{1}{s(u',v)} du',$$

and use the previous bound to get $\partial_u\big(|\psi| e^{-\varphi}\big) \le -(b+1)\partial_u e^{-\varphi}$. Integrating this we obtain:

$$|\psi(u,v)| \le |\psi(-2a,v)| e^{\varphi(u,v)} + (b+1)\big(e^{\varphi(u,v)} - 1\big), \tag{46}$$

provided that $(C1)-(C2)$ hold for all (u',v) with $u' \in [-2a,u]$. To obtain a bound for ψ, we firstly need to estimate φ. For this we estimate $s(u,v)$.

Recall that by (32) we have $\mu(u,v) \le \mu(u,0) + c_1 s \log \frac{1}{s}$, provided that (C1) holds. This implies that $\mu(u,v) \le \frac{1}{3} + \frac{2}{3}\mu(u,0)$, if we also assume $c_1 s \log \frac{1}{s} \le \frac{1}{3\varkappa(u,0)}$. Using Lemma 2, this condition is implied by:

$$s(u,v) \log \frac{1}{s(u,v)} \le \frac{e^{-\gamma(u)}}{6c_1 \varkappa(-2a,0)} \tag{C3}$$

We note this is strictly stronger than (C2). Thus, if $(C1)-(C3)$ hold, then:

$$\varkappa(u,v) \le \frac{3}{2}\varkappa(u,0) \tag{47}$$

From (15) we get that if $(C1)-(C3)$ hold for all (u',v) with $u' \in [-2a,u]$, then $\log(-2\nu)(u',v) \geq (\frac{3}{2}\varkappa(u',0)-1)s(u',v)$. Using this in $-u\partial_u s = 1-\nu e^{-s}$ we obtain:

$$\partial_u s(u',v) \geq -\frac{1}{u'} + \frac{1}{u'} e^{(\frac{3}{2}\varkappa(u',0)-2)s(u',v)} \tag{48}$$

This lower bound on $\partial_u s$ allows us control $s(u',v)$ by $s(u,v)$ when $u' < u$. Indeed, after some work one obtains by integrating (48) that:

$$s(u',v) \leq 2s(u,v)\left(\frac{u}{u'}\right)^{1/2} \exp\frac{3}{2}\big(\gamma(u)-\gamma(u')\big), \tag{49}$$

provided that $(C1)-(C3)$ hold for all (u',v) with $u' \in [-2a,u]$, and:

$$s(u,v) \leq \frac{e^{-\frac{3}{2}\gamma(u)}}{2c_3\varkappa(-2a,0)}, \tag{C4}$$

holds at (u,v), where $c_3 = \frac{19}{2}(e-2)$. Using the inequality $x^{1/2}\log\frac{1}{x} \leq \frac{2}{e}$ for $x>0$ small, we get:

$$s(u',v)\log\frac{1}{s(u',v)} \leq \frac{2^{3/2}}{e} s(u,v)^{1/2}\left(\frac{u}{u'}\right)^{1/4} \exp\frac{3}{4}\big(\gamma(u)-\gamma(u')\big)$$

This is the estimate we need in order to bound φ. We have that:

$$\int_{-2a}^{u} \frac{1}{-u'} e^{2\gamma(u')} s(u',v)\log\frac{1}{s(u',v)} du' \leq \frac{2^{3/2}}{e} s(u,v)^{1/2} e^{2\gamma(u)} \int_{-2a}^{u} \frac{1}{-u'}\left(\frac{u}{u'}\right)^{1/4} du' \leq \frac{2^{7/2}}{e} s(u,v)^{1/2} e^{2\gamma(u)}.$$

Therefore, we proved that for $c_4 = 2^{7/2}/e \cdot c_2$, we have:

$$\varphi(u,v) \leq c_4 \varkappa(-2a,0)^2 e^{2\gamma(u)} s(u,v)^{1/2}, \tag{50}$$

provided that $(C1)-(C3)$ hold for all (u',v) with $u' \in [-2a,u]$, and (C4) holds at (u,v).

We now use inequality (46) to bound ψ. We introduce the condition:

$$s(u,v) \leq \min\left(\log 2, \frac{h}{48(b+1)}\right)^2 \frac{e^{-4\gamma(u)}}{c_4^2\varkappa(-2a,0)^4} \tag{C5}$$

which together with the above assumptions implies that:

$$e^{\varphi(u,v)} \leq 2,\ \ e^{\varphi(u,v)}-1 \leq 2\varphi(u,v) \leq \frac{h}{48(b+1)}$$

Finally, we also assume that for some $v_0 < \epsilon$:

$$\sup_{v\in[0,v_0]} |\psi(-2a,v)| \leq \frac{h}{48} \tag{C6}$$

We get by (46) that if $(C1)-(C3)$ hold for all (u',v_0) with $u' \in [-2a,u]$, (C4),(C5) hold at (u,v_0) and (C6) holds at v_0, then:

$$|\psi(u,v_0)| \leq \frac{h}{12} \tag{51}$$

Remark 10. *If all conditions needed above hold, i.e.* ($C1$) $-$ ($C3$) *hold for all* (u', v_0) *with* $u' \in [-2a, u]$, *(C4),(C5) hold at* (u, v_0) *and (C6) holds at* v_0*, we say that "all conditions hold for* (u, v_0)*" for brevity.*

Step 3. Formation of a trapped surface

We set out to prove that $\frac{\theta}{\lambda}$ is large in a small region to the exterior of C_0^-, expecting this will lead to the formation of a trapped surface because of formula (35). We notice we can rewrite the definition of ψ as:

$$\frac{\theta}{\lambda}(u,v) \cdot \frac{r(u,v)}{r(u,0)} = \frac{\theta}{\lambda}(u,0) + \psi(u,v)e^{\gamma(u)}.$$

Recall that $\frac{\theta}{\lambda}(u,0)$ is large according to (37), as a direct consequence of the infinite blue-shift and the fact that the problem is not finely tuned, i.e. condition (36). Moreover, we proved in the previous step that $|\psi|$ is small. Thus, $\frac{\theta}{\lambda}(u,0)$ is the dominant term in the above, and we have that for a sequence $u_n \to 0-$:

$$\left| \frac{\theta}{\lambda}(u_n, v_0) \cdot \frac{r(u_n, v_0)}{r(u_n, 0)} \right| \geq \frac{h}{4} e^{\gamma(u_n)}, \tag{52}$$

if all conditions hold for (u_n, v_0). We notice that $\partial_v s > 0$, so the functions s, $s \log \frac{1}{s}$ are increasing in v, for $s \leq c_0$ small enough. Thus, if all conditions hold for (u, v_0), then all conditions hold for (u, v), for all $v \in [0, v_0]$. We deduce that if all conditions hold for (u_n, v_0), then for all $v \in [0, v_0]$:

$$\left| \frac{\theta}{\lambda}(u_n, v) \cdot \frac{r(u_n, v)}{r(u_n, 0)} \right| \geq \frac{h}{4} e^{\gamma(u_n)}, \tag{53}$$

This is the desired quantitative estimate of the fact that $\frac{\theta}{\lambda}$ is large in a small region to the exterior of C_0^-. We use it in (35) to get that for any $v \in [0, v_0]$:

$$\eta(u_n, v) = \frac{1}{r(u_n, v)} \int_0^v \frac{1}{\varkappa} \cdot \frac{\theta^2}{\lambda}(u_n, v')dv' \geq \frac{r(u_n,0)^2}{r(u_n,v)} \cdot \frac{h^2}{16} \cdot e^{2\gamma(u_n)} \int_0^v \frac{1}{\varkappa(u_n, v')} \cdot \frac{\lambda}{r^2}(u_n, v')dv' \geq$$

$$\geq \frac{h^2}{64\varkappa(-2a,0)} \cdot e^{\gamma(u_n)} \cdot \frac{r(u_n,0)}{r(u_n,v)} \left(1 - \frac{r(u_n,0)}{r(u_n,v)}\right) \geq \frac{c_5 h^2}{\varkappa(-2a,0)} \cdot e^{\gamma(u_n)} \cdot s(u_n, v),$$

where $c_5 = \frac{1}{64} \cdot e^{-c_0}(1 - e^{-c_0})/c_0$, and we used the fact that the function $e^{-s}(1-e^{-s})/s$ is decreasing. We want this to give that η is large enough so we can apply Theorem 8 to obtain that a trapped surface will form. We take any constant $c_6 < c_5/c_1$ and we assume that:

$$\exists n \in \mathbb{N},\ v \in [0, v_0],\ \text{such that } s(u_n, v) = \exp\left(\frac{-c_6 h^2 e^{\gamma(u_n)}}{\varkappa(-2a,0)}\right) \tag{C7}$$

If all conditions hold for this (u_n, v), we have:

$$\eta(u_n, v) < c_1 s(u_n, v) \log \frac{1}{s(u_n, v)} = \frac{c_1 c_6 h^2}{\varkappa(-2a,0)} \cdot e^{\gamma(u_n)} \cdot s(u_n, v) < \eta(u_n, v),$$

so the lower bound that we obtained on η contradicts the condition to not form trapped surfaces (32).

Step 4. Region to the exterior of C_0^-

We recall that in order to obtain the above contradiction, we assumed that certain conditions on r hold in a small region to the exterior of C_0^-. We assumed that (C7) holds, and for those $n \in \mathbb{N}$, $v \in [0, v_0]$ we have that $(C1) - (C3)$ hold for all (u', v) with $u' \in [-2a, u]$, (C4),(C5) hold at (u, v) and (C6) holds at v. We now prove all this conditions are consistent.

By convention we have that α is right continuous on C_0^+, so ψ is also right continuous. For $\epsilon > 0$ small enough we have:

$$\sup_{v \in [0,\epsilon)} |\psi(-2a, v)| \leq \frac{h}{48},$$

so condition (C6) hold at any $v_0 \in [0, \epsilon)$.

We prove that (C7) holds. Suppose that there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$s(u_n, v_0) < S_n := \exp\left(\frac{-c_6 h^2 e^{\gamma(u_n)}}{\varkappa(-2a, 0)}\right).$$

Since $\gamma(u_n)$ is unbounded, we get that for all n large enough conditions $(C4) - (C5)$ hold at (u_n, v_0). We also have that conditions $(C1) - (C3)$ hold at (u_n, v_0). We define:

$$u_* = \inf\left\{u \in [-2a, u_n] : \ (C1) - (C3) \text{ hold at } (u', v_0) \text{ for all } u' \in [u, u_n]\right\}$$

All the necessary conditions to apply (49) on $[u_*, u_n]$ hold, so we obtain:

$$s(u_*, v_0) \leq 2s(u_n, v_0)\left(\frac{u_n}{u_*}\right)^{1/2} \exp\frac{3}{2}\big(\gamma(u_n) - \gamma(u_*)\big) \leq 2s(u_n, v_0)\exp\frac{3}{2}\gamma(u_n)$$

We need to consider the following three cases:

- If $u_* = -2a$. For all n large enough we obtain the contradiction:

$$s(u_n, v_0) \geq \frac{1}{2}s(-2a, v_0)e^{-\frac{3}{2}\gamma(u_n)} > S_n.$$

- If at u_* we have equality in (C1), so $s(u_*, v_0) = c_0$. For all n large enough we obtain the contradiction:

$$s(u_n, v_0) \geq \frac{1}{2}c_0 e^{-\frac{3}{2}\gamma(u_n)} > S_n.$$

- If at u_* we have equality in (C3) (note that (C2) is strictly weaker). We have that:

$$\frac{2}{e}s(u_*, v_0)^{1/2} \geq s(u_*, v_0)\log\frac{1}{s(u_*, v_0)} = \frac{e^{-\gamma(u_*)}}{6c_1\varkappa(-2a, 0)} \geq \frac{e^{-\gamma(u_n)}}{6c_1\varkappa(-2a, 0)}.$$

  For all n large enough we obtain the contradiction:

$$s(u_n, v_0) \geq \frac{1}{2}s(u_*, v_0)e^{-\frac{3}{2}\gamma(u_n)} \geq \left(\frac{e}{12\sqrt{2}c_1\varkappa(-2a, 0)}\right)^2 e^{-\frac{7}{2}\gamma(u_n)} > S_n.$$

Therefore, we proved that there exists an infinite subsequence $u_n \to 0-$ such that $S_n \in [0, s(u_n, v_0)]$. We also obtain a sequence $v_n \in [0, v_0]$ such that $S_n = s(u_n, v_n)$. Thus, (C7) holds for all n large enough and v_n.

Finally, we prove that for n large enough all conditions hold for (u_n, v_n). Indeed, since γ_n is unbounded we have that (C4),(C5) hold at (u_n, v_n), and also that $(C1) - (C3)$ hold at (u_n, v_n). We define:

$$u_* = \inf \left\{ u \in [-2a, u_n] : \ (C1) - (C3) \text{ hold at } (u', v_n) \text{ for all } u' \in [u, u_n] \right\}$$

We can apply (49) on $[u_*, u_n]$ to get:

$$s(u_*, v_n) \leq 2s(u_n, v_n) \left(\frac{u_n}{u_*} \right)^{1/2} \exp \frac{3}{2} \big(\gamma(u_n) - \gamma(u_*) \big) \leq 2s(u_n, v_n) \exp \frac{3}{2} \gamma(u_n)$$

Taking n large enough we cannot have equality in any of $(C1) - (C3)$, so we conclude that $u_* = -2a$. Thus, $(C1) - (C3)$ hold at (u', v_n) for all $u' \in [-2a, u_n]$, so we proved that all conditions hold for (u_n, v_n). $\square$

Remark 11. *The conclusion of the theorem holds if we replace the condition that the data is AC with the condition that the data is BV and γ is unbounded. However, since the extension principle in Theorem 7 only holds for AC data, we can have a BV solution with γ bounded that cannot be extended to the future of C_0^- at the center. In this case the instability mechanism we want to exploit is not present, so we cannot hope to prove the Weak Cosmic Censorship conjecture for BV data with this approach.*

6.2 Weak Cosmic Censorship

In the previous sections we saw that in order to allow for naked singularities, solutions must satisfy certain restrictive properties. In particular, we have the following necessary conditions: $b_\Gamma \neq i^+$, $\mathcal{A} = \emptyset$, $\gamma(u) \to \infty$ as $u \to 0-$, and

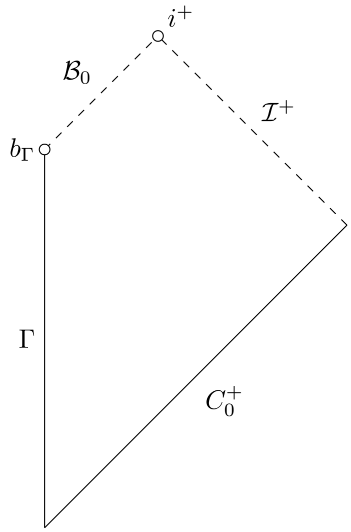
$$\lim_{u \to 0-} I(u) = \frac{\theta}{\lambda}(-2a, 0).$$

To complete the proof of the Weak Cosmic Censorship, we want to prove that for generic integrable absolutely continuous initial data the above conditions do not hold. In this section, we only prove that integrable AC initial data can be perturbed in the integrable BV class to a solution with complete future null infinity. We explain at the end what needs to be done to prove the Weak Cosmic Censorship.

The well-posedness result in the class of BV initial data implies that a solution with AC initial data is determined by:

$$\vartheta = \frac{\theta}{\lambda}\Big|_{u=-2a} \in AC\big([-a_0, v_{\mathcal{I}^+})\big) \cap L^1\big([-a_0, v_{\mathcal{I}^+})\big).$$

We define the exceptional set $\mathcal{E}$ to consist of initial data $\vartheta \in AC\big([-a_0, v_{\mathcal{I}^+})\big) \cap L^1\big([-a_0, v_{\mathcal{I}^+})\big)$ such that in the maximal globally hyperbolic future development we have that $b_\Gamma \neq i^+$ and $\mathcal{A} = \emptyset$. Such a spacetime has the Penrose diagram:



We prove the following:

Theorem 11. *For any $\vartheta \in \mathcal{E}$ there exists a one-dimensional linear subspace $\Pi_\vartheta \subset BV\big([-a_0, v_{\mathcal{I}^+})\big) \cap L^1\big([-a_0, v_{\mathcal{I}^+})\big)$ such that:*

- *The maximal development of any $\tilde{\vartheta} \in \Pi_\vartheta \backslash \{\vartheta\}$ has $b_\Gamma \neq i^+$ and $\mathcal{A} \neq \emptyset$,*
- *$\vartheta \neq \vartheta' \in \mathcal{E}$ implies $\Pi_\vartheta \cap \Pi_{\vartheta'} = \emptyset$.*

Proof. By Theorem 10 we know that $\vartheta \in \mathcal{E}$ implies $\lim_{u\to 0-} I(u) = \theta/\lambda(-2a, 0) = \vartheta(0)$. Let $f : \mathbb{R} \to [0, \infty)$ be an integrable function, vanishing on $(-\infty, 0)$, such that $f|_{[0,\infty)}$ is absolutely continuous and $\lim_{v\to 0+} f(v) = 1$. This function is BV but not AC on $[-a_0, v_{\mathcal{I}^+})$.

For any $\lambda \in \mathbb{R}$ we consider initial data:

$$\tilde{\vartheta}_\lambda = \vartheta + \lambda f \in BV\big([-a_0, v_{\mathcal{I}^+})\big) \cap L^1\big([-a_0, v_{\mathcal{I}^+})\big). \tag{54}$$

This defines a one dimensional linear subspace of initial data:

$$\Pi_\vartheta = \{\tilde{\vartheta}_\lambda : \ \lambda \in \mathbb{R}\} \subset BV\big([-a_0, v_{\mathcal{I}^+})\big) \cap L^1\big([-a_0, v_{\mathcal{I}^+})\big).$$

By our construction, we have that $\tilde{\vartheta}_\lambda|_{[-a_0,0)} = \vartheta|_{[-a_0,0)}$, so according to the uniqueness theorem the two solutions coincide in $\mathcal{D}(-2a, 0)$, i.e. in the interior of C_0^-. Since μ is continuous, we get that $\mu_\lambda = \mu$ on C_0^-. By the extension principle in Theorem 7 we have that $\mu \not\to 0$ on C_0^-, so the development of $\tilde{\vartheta}_\lambda$ cannot be extended at the center to the future of C_0^-. In particular, we also have that $b_\Gamma \neq i^+$ for the perturbed solution. We then get that $\gamma_\lambda(u) = \gamma(u)$ and $I_\lambda(u) = I(u)$. We notice that γ_λ is unbounded for the perturbed solution. We take $v \to 0+$ in (54):

$$\tilde{\vartheta}_\lambda(0) = \vartheta(0) + \lambda \lim_{v\to 0+} f(0) = \lim_{u\to 0-} I(u) + \lambda.$$

Therefore, if $\lambda \neq 0$ we have that:

$$\tilde{\vartheta}_\lambda(0) \neq \lim_{u \longrightarrow 0-} I(u) = \lim_{u \longrightarrow 0-} I_\lambda(u).$$

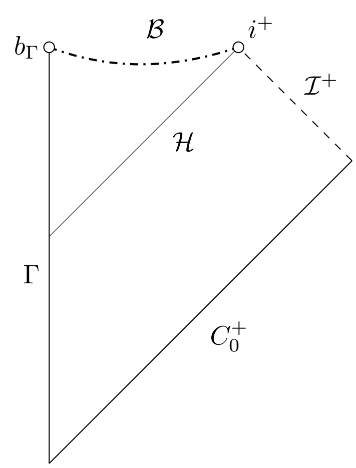
But we remarked that Theorem 10 holds for BV solutions if we also assume that γ_λ is unbounded. Therefore, the perturbed solution has $\mathcal{A} \neq \emptyset$, proving the first part of the theorem.

Consider now $\vartheta, \vartheta' \in \mathcal{E}$, and suppose that $\tilde{\vartheta}_\lambda = \tilde{\vartheta'}_{\lambda'}$ for some $\lambda, \lambda' \in \mathbb{R}$. Again, since f vanishes on $(-\infty, 0)$ we get that $\vartheta|_{[-a_0,0)} = \tilde{\vartheta}_\lambda|_{[-a_0,0)} = \tilde{\vartheta'}_{\lambda'}|_{[-a_0,0)} = \vartheta'|_{[-a_0,0)}$. According to the uniqueness theorem, the two solutions corresponding to ϑ and ϑ' coincide in the interior of C_0^-. Since μ is continuous, we get that $\mu = \mu'$ on C_0^-, so $\gamma(u) = \gamma'(u)$ and $I(u) = I'(u)$. We obtain:

$$\lim_{u \longrightarrow 0-} I(u) + \lambda = \tilde{\vartheta}_\lambda(0) = \tilde{\vartheta'}_{\lambda'}(0) = \lim_{u \longrightarrow 0-} I'(u) + \lambda',$$

which gives $\lambda = \lambda'$. Finally, this implies $\vartheta = \vartheta'$. □

We proved that we can perturb any solution with integrable AC initial data that has a naked singularity to a BV solution with the Penrose diagram:



The second conclusion of the theorem implies that the exceptional set has positive co-dimension in the space of initial data $BV\big([-a_0, v_{\mathcal{I}^+})\big) \cap L^1\big([-a_0, v_{\mathcal{I}^+})\big)$. This does not settle the Weak Cosmic Censorship, because the perturbation is less regular than the original spacetime. There are two natural approaches to address this issue:

1. We could attempt to prove the Weak Cosmic Censorship conjecture directly in the less regular class of BV solutions. However, as we mentioned above, the blue-shift instability is not necessarily present for BV solutions, so we would need an entirely different proof in the case when γ is bounded on C_0^-.

2. We could commit to proving Weak Cosmic Censorship for AC initial data, so we should only perturb the initial data by AC functions. It is unclear how to do this using the instability theorem 10. However,

solutions with naked singularities satisfy even more restrictive properties than the conditions of our instability theorem require, so one could look for more specialized instability results.

The second approach was used by Christodoulou in [6]. He carried out a detailed analysis in the case $\lim_{u\to 0-} I(u) = \theta/\lambda(-2a,0)$, which led to an even more restrictive condition for the spacetime to not form trapped surfaces. In the context of this stronger instability result, any integrable AC initial data leading to a solution with naked singularities can be perturbed by an AC function, so that the obtained spacetime has a trapped surface. He established the following:

Theorem 12 (Weak Cosmic Censorship)**.** *For any $\vartheta \in \mathcal{E}$ there exists a one-dimensional linear subspace $\Pi_\vartheta \subset AC\big([-a_0, v_{\mathcal{I}^+})\big) \cap L^1\big([-a_0, v_{\mathcal{I}^+})\big)$ such that:*

- $\big(\Pi_\vartheta \backslash \{\vartheta\}\big) \cap \mathcal{E} = \emptyset$,
- $\vartheta \neq \vartheta' \in \mathcal{E}$ *implies* $\Pi_\vartheta \cap \Pi_{\vartheta'} = \emptyset$.

Thus, $\mathcal{E}$ has co-dimension at least one in the space of initial data $AC\big([-a_0,\infty)\big) \cap L^1\big([-a_0,\infty)\big)$.

Remark 12 (Strong Cosmic Censorship)**.** *We can relax the definition of the exceptional set $\mathcal{E}$ and ask that it consists of initial data $\vartheta \in AC\big([-a_0, v_{\mathcal{I}^+})\big) \cap L^1\big([-a_0, v_{\mathcal{I}^+})\big)$ such that in the maximal globally hyperbolic future development we have that $b_\Gamma \neq i^+$ and b_Γ is not a limit point of $\mathcal{A}$. Such spacetimes can have an outgoing Cauchy horizon arising from b_Γ. Our previous arguments apply in this case too, since we only assumed that there is no marginally trapped surface in a neighbourhood of C_0^-. Thus, the theorem cited above also holds for this definition of the exceptional set. We notice that for data in $\Pi_\vartheta \backslash \{\vartheta\}$, the maximal globally hyperbolic future development has $\mathcal{B}_0 = b_\Gamma$, so in particular it has no Cauchy horizon. This proves that, generically, spherically symmetric solutions of the Einstein scalar field equations do not have Cauchy horizons, establishing the Strong Cosmic Censorship conjecture.*

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