Nonlinear Scattering Theory for Asymptotically de Sitter Vacuum Solutions in All Even Spatial Dimensions

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Abstract

In this dissertation, we establish a definitive quantitative nonlinear scattering theory for asymptotically de Sitter solutions of the Einstein vacuum equations in $(n+1)$ dimensions with $n \geq 4$ even, which are determined by small scattering data at the spacelike asymptotic boundaries $\mathcal{I}^-$ and $\mathcal{I}^+$. The case of even spatial dimension n poses significant challenges compared to its odd counterpart and was left open by the previous works in the literature. Here, scattering theory is understood to mean existence and uniqueness of scattering states, asymptotic completeness, and the existence of an invertible scattering map with quantitative control on its norm. The existence and uniqueness of scattering states imply that for any small asymptotic data there exists a unique global solution to the Einstein equations, which remains close to the de Sitter metric. Asymptotic completeness is the converse statement, showing that any such solution induces asymptotic data at $\mathcal{I}^-$ and at $\mathcal{I}^+$. For sufficiently small asymptotic data, we construct the scattering map $\mathcal{S}$ taking data at $\mathcal{I}^-$ to data at $\mathcal{I}^+$, and we show that the map $\mathcal{S}$ is locally invertible and locally Lipschitz at the de Sitter data, with respect to a Sobolev-type norm. The scattering map result is sharp and avoids any "derivative loss", in the sense that we measure the smallness of asymptotic data at $\mathcal{I}^-$ and $\mathcal{I}^+$ using the same Sobolev norm. The proof of the sharp result requires a detailed analysis of the Einstein equations involving a geometric Littlewood-Paley decomposition of the solution.

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Contents

Chapter 1

Introduction

In this thesis, we aim to complete the understanding of the nonlinear scattering theory for $(n+1)$-dimensional asymptotically de Sitter vacuum solutions determined by small scattering data on a suitably defined asymptotic boundary $\mathcal{I}^{\pm}$. The case $n=3$ was proved by Friedrich in [Fri83, Fri86], while the case of all $n \geq 3$ odd was proved by Anderson in [And05]. In this work, we treat the case of even spatial dimension $n \geq 4$, which contains significant new challenges and was left open by the previous works in the literature.

Asymptotically de Sitter vacuum solutions. For any $n \geq 3$, we consider the $(n+1)$-dimensional Einstein vacuum equations with positive cosmological constant $\Lambda = \frac{n(n-1)}{2}$:

$$Ric_{\mu\nu} - \frac{1}{2} R \widetilde{g}_{\mu\nu} + \Lambda \widetilde{g}_{\mu\nu} = 0. \tag{1.1}$$

The ground state solution of (1.1) is given by the $(n+1)$-dimensional de Sitter space $\big(\mathbb{R} \times S^n, \widetilde{g}_{dS}\big)$:

$$\widetilde{g}_{dS} = -dT^2 + \cosh^2(T) \cdot \not{g}_{S^n}, \tag{1.2}$$

where $\not{g}_{S^n}$ denotes the standard round metric on S^n. The solution $\big(\mathbb{R} \times S^n, \widetilde{g}_{dS}\big)$ represents the higher dimensional generalization of the metric introduced in [dS17]. We denote past infinity $\{T \to -\infty\}$ by $\mathcal{I}^-$, and future infinity $\{T \to \infty\}$ by $\mathcal{I}^+$. Both $\mathcal{I}^-$ and $\mathcal{I}^+$ can be identified with S^n and can be understood as asymptotic boundaries of the spacetime.

Due to the hyperbolic nature of the Einstein vacuum equations (1.1), we study dynamical solutions of (1.1) obtained by solving an initial value problem. We review briefly the standard setting of Cauchy initial data prescribed on a spacelike hypersurface. The data consist of a Riemannian

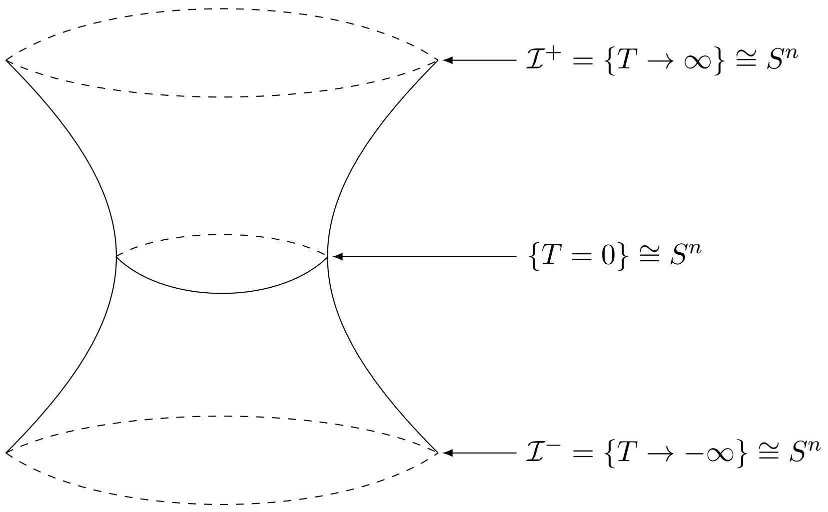

Figure 1.1: Diagram of the $(n+1)$-dimensional de Sitter space $\big(\mathbb{R} \times S^n, \widetilde{g}_{dS}\big)$.

manifold $\big(S^n, g_0\big)$ and a symmetric 2-tensor k_0, which satisfy certain constraint equations. In the spacetime obtained by solving (1.1) locally with the given initial data, $\big(S^n, g_0\big)$ embeds as the initial data hypersurface with second fundamental form given by k_0. Moreover, the constraint equations are given by the Gauss and Codazzi equations for g_0 and k_0.

In the current work, we will focus instead on solutions of (1.1) arising from scattering data prescribed at infinity (at $\mathcal{I}^-$ or $\mathcal{I}^+$). In this setting, scattering data consist of a Riemannian manifold $\big(S^n, \slashed{g}_0\big)$ and a symmetric traceless 2-tensor $\check{h}$ on S^n, which satisfies an additional constraint called the straightness condition. We briefly describe the interpretation of scattering data and the differences to the setting of Cauchy initial data. We consider the case of data prescribed at $\mathcal{I}^-$ for the purpose of exposition. Similarly to the case of Cauchy initial data, $\slashed{g}_0$ represents a suitable limit at $\mathcal{I}^-$ of the Riemannian metrics induced by the spacetime metric on the spheres S^n at early times. However, it turns out that a similar attempt to consider the limit of the second fundamental form of the spheres S^n at early times yields a tensor that is completely determined by $\slashed{g}_0$. Instead, $\check{h}$ represents a higher order term present in the expansion of the spacetime metric at $\mathcal{I}^-$, as we shall explain in detail below.

Solutions of (1.1) determined by scattering data at infinity are called asymptotically de Sitter spaces ([And05]). The local well-posedness theory for scattering data at infinity is proved in [RSR18] (see also [FG85, Ren04, FG12, Hin24]), with asymptotic data given by $\slashed{g}_0$ and $\check{h}$ as above. We also assume that the data are smooth for simplicity. For any such data, we obtain in a neighborhood of $\mathcal{I}^-$ a unique solution of the form:

$$\widetilde{g} = -dT^2 + e^{-2T}\widetilde{\slashed{g}}_{AB}(T, \theta^1, \ldots, \theta^n)d\theta^A d\theta^B, \tag{1.3}$$

where we denote by $\{\theta^A\}$ coordinates associated to an arbitrary chart on S^n. As before, in (1.3) we present the case of scattering data at past infinity $\mathcal{I}^-$ and we note that the same definitions apply for scattering data at $\mathcal{I}^+$, upon replacing T by $-T$.

The expansion at $\mathcal{I}^-$ in the even n case. The relation between the solution $\widetilde{g}$ and the scattering data is seen in the expansion satisfied by the rescaled metric $\widetilde{\slashed{g}}$ induced on $\{T\} \times S^n$ as $T \to -\infty$. For all $n \geq 4$ even and for all $T < 0$ small enough, we have in a Lie-propagated frame:

$$\widetilde{\slashed{g}} = \slashed{g}_0 + \frac{e^{2T}}{2^2}\slashed{g}_1 + \ldots + \frac{e^{(n-2)T}}{2^{n-2}(n/2-1)!}\slashed{g}_{n/2-1} + \frac{2Te^{nT}}{2^n(n/2)!}\mathcal{O} + \frac{e^{nT}}{2^n(n/2)!}\check{k} + O\big(Te^{(n+2)T}\big), \tag{1.4}$$

where the tensors $\slashed{g}_1, \ldots, \slashed{g}_{n/2-1}, \mathcal{O}, \mathrm{tr}\check{k}$ are determined by $\slashed{g}_0$ via certain compatibility relations, $\check{h}$ is the trace free part of $\check{k}$, and the higher order terms in the expansion are determined by $\slashed{g}_0$ and $\check{h}$.

The main challenge present in the case of even spatial dimension $n \geq 4$ compared to its odd counterpart can be seen in the expansion (1.4). This represents the Fefferman-Graham expansion introduced in [FG85], also playing a fundamental role in [Ren04, FG12, RSR18], as we explain in Section 1.1. We point out that for $n \geq 4$ even the expansion is not smooth at $\mathcal{I}^-$ in terms of the Fefferman-Graham coordinate $\tau = e^T/2$ due to the presence of the term $\mathcal{O}$, called the obstruction tensor. On the other hand, the obstruction tensor vanishes identically in the odd n case leading to a smooth expansion of the solution in τ at $\mathcal{I}^-$.

From the dynamical point of view, the de Sitter metric (1.2) is the unique solution of (1.1) with scattering data at $\mathcal{I}^-$ given by $\big(\slashed{g}_0\big)_{dS} = \frac{1}{4}\slashed{g}_{S^n}$ and $\check{h}_{dS} = 0$. It is also smooth at $\mathcal{I}^-$, since the obstruction tensor $\mathcal{O}_{dS}$ vanishes, as can be seen from its Fefferman-Graham expansion:

$$\widetilde{\slashed{g}}_{dS} = \frac{1}{4}\slashed{g}_{S^n} + \frac{e^{2T}}{2}\slashed{g}_{S^n} + \frac{e^{4T}}{4}\slashed{g}_{S^n}.$$

The main result: a complete scattering theory for $n \geq 4$ even. We study the global in time behavior of asymptotically de Sitter solutions of (1.1) for $n \geq 4$ even, determined by scattering data close to the data of de Sitter space with respect to some suitable norm. Our main result is establishing a complete scattering theory for such solutions. We notice that the corresponding small data result in the setting of Cauchy initial data consists of proving global existence and orbital stability, see [Rin08]. The additional difficulties that we encounter in the case of the scattering problem are the need to evolve the data from past infinity as opposed to from a Cauchy hypersurface, and the need to recover the scattering data at future infinity, which requires sharp control of the solution at higher order despite the lack of smoothness in time caused by the obstruction tensor $\mathcal{O}$.

In order to state the main result, we briefly introduce the notions of asymptotic initial data set, asymptotic initial data norm, and asymptotically de Sitter vacuum solutions determined by small data. We refer the reader to Remark 1.3 and Definition 9.1 for the precise definitions.

Given smooth scattering data $\big(\slashed{g}_0, \check{h}\big)$, we define the corresponding *asymptotic initial data set* $\widetilde{\Sigma}\big(\slashed{g}_0, \check{h}\big)$ to be the collection of tensors on S^n consisting of: the metric $\slashed{g}_0$; the tensors $\slashed{g}_1, \dots, \slashed{g}_{n/2-1}, \mathrm{tr}\check{k}$ defined by $\slashed{g}_0$ using the compatibility relations, together with certain angular derivatives of these tensors; the obstruction tensor $\mathcal{O}$ defined by $\slashed{g}_0$ using the compatibility relations; and the renormalized tensor $\check{\mathfrak{h}} = \check{h} - 2\big(\log \nabla\big)\mathcal{O}$, where the operator $\log \nabla$ is defined using the geometric Littlewood-Paley decomposition in Section 6. We point out that the surprising need to renormalize $\check{h}$ is related to the lack of smoothness of the expansion (1.4), and again poses difficulties in our problem.

We also define the *asymptotic initial data norm*, which measures closeness to the de Sitter data. For any tensor $\phi \in \widetilde{\Sigma}\big(\slashed{g}_0, \check{h}\big)$, we denote by $\phi_* = \phi - \phi_{dS}$ the tensor obtained as the difference of ϕ and its de Sitter value. For any $M > 0$, we define the *asymptotic initial data norm of order* M by:

$$\Big\|\widetilde{\Sigma}\big(\slashed{g}_0, \check{h}\big)\Big\|_M^2 = \sum_{\phi \in \widetilde{\Sigma}(\slashed{g}_0, \check{h})} \big\|\phi_*\big\|_{H^{M+1}(S^n)}^2, \tag{1.5}$$

where $H^{M+1}(S^n)$ represents the Sobolev norm on S^n with respect to the metric $\slashed{g}_0$.

We denote by $\widetilde{\Sigma}_{dS} = \widetilde{\Sigma}\big(\frac{1}{4}\slashed{g}_{S^n}, 0\big)$ the initial data set corresponding to de Sitter space. For any $\epsilon > 0$, we define the set of smooth ϵ-small asymptotic data of order M by:

$$B_\epsilon^M\big(\widetilde{\Sigma}_{dS}\big) = \Big\{\widetilde{\Sigma}\big(\slashed{g}_0, \check{h}\big) : \ \big\|\widetilde{\Sigma}\big(\slashed{g}_0, \check{h}\big)\big\|_M < \epsilon\Big\}. \tag{1.6}$$

We define *asymptotically de Sitter vacuum solutions determined by small data* to be the solutions of (1.1) with ϵ-small asymptotic data of order M.

Using these definitions, we state the main result of the thesis:

Theorem 1.1. *For any even integer $n \geq 4$, we have a complete scattering theory for $(n+1)$-dimensional asymptotically de Sitter vacuum solutions determined by small data. For any $M > 0$ large enough there exists $\epsilon_0 > 0$ small enough, such that for any $0 < \epsilon \leq \epsilon_0$ we have:*

1. ***Existence and uniqueness of scattering states:*** *for any ϵ-small asymptotic data of order M at $\mathcal{I}^-$ or $\mathcal{I}^+$ given by $\widetilde{\Sigma}\big(\not{g}_0, \check{h}\big) \in B^M_\epsilon\big(\widetilde{\Sigma}_{dS}\big)$, there exists a unique smooth global solution $\big(\widetilde{\mathcal{M}}, \widetilde{g}\big)$ of the form (1.3) to the Einstein vacuum equations (1.1) which remains quantitatively close to the de Sitter metric and can be represented by a diagram similar to Figure 1.1;*

2. ***Asymptotic completeness:*** *any smooth solution of the Einstein vacuum equations (1.1) of the form (1.3), which is quantitatively close to the de Sitter metric at a finite time T, exists globally and induces scattering data $\big(\not{g}_0, \check{h}\big)$ at $\mathcal{I}^-$ and $\big(\underline{\not{g}_0}, \underline{\check{h}}\big)$ at $\mathcal{I}^+$;*

3. ***Existence of a scattering map with quantitative estimates:*** *there exists a constant $C_M > 0$ independent of ϵ, such that we have a well-defined scattering map taking asymptotic data at $\mathcal{I}^-$ to asymptotic data at $\mathcal{I}^+$:*

$$\begin{cases} \mathcal{S} : B^M_\epsilon\big(\widetilde{\Sigma}_{dS}\big) \to B^M_{C_M\epsilon}\big(\widetilde{\Sigma}_{dS}\big), \\ \mathcal{S}\Big(\widetilde{\Sigma}\big(\not{g}_0, \check{h}\big)\Big) = \widetilde{\Sigma}\big(\underline{\not{g}_0}, \underline{\check{h}}\big). \end{cases} \tag{1.7}$$

The scattering map is locally invertible and locally Lipschitz at $\widetilde{\Sigma}_{dS}$, in the sense that it satisfies the quantitative estimate:

$$\Big\| \mathcal{S}\big(\widetilde{\Sigma}(\not{g}_0, \check{h})\big) \Big\|_M \leq C_M \Big\| \widetilde{\Sigma}(\not{g}_0, \check{h}) \Big\|_M. \tag{1.8}$$

Remark 1.1. *The result for the scattering map $\mathcal{S}$ is sharp and avoids any "derivative loss", in the sense that in* (1.7) *and* (1.8) *we use the same Sobolev-type asymptotic initial data norm of order M to measure the smallness of asymptotic data at $\mathcal{I}^-$ and $\mathcal{I}^+$.*

Remark 1.2. *We state Theorem 1.1 for smooth scattering data $\not{g}_0, \check{h}$ for simplicity of exposition, but since we only make quantitative assumptions at the level of the asymptotic initial data norm*

(1.5)*, our result can be extended by density to the case of finite regularity scattering data which is small in the sense of* (1.6)*.*

In the remainder of the introduction, we flesh out the previous discussion with more details. In Section 1.1 we discuss some relevant previous results. In Section 1.2 we introduce the ambient metric formulation of the problem and restate our main result in an equivalent form. In Section 1.3 we discuss the ideas of the proof in some detail. Finally, in Section 1.4 we outline the structure of the rest of the thesis.

1.1 Previous Results

We present some previous results relevant for the scattering theory of asymptotically de Sitter vacuum solutions.

1.1.1 The Stability of de Sitter Space

Friedrich proved in [Fri83, Fri86] that $(3+1)$-dimensional de Sitter space is non-linearly stable to small perturbations of the asymptotic data at $\mathcal{I}^-$. The proof uses the key fact that in $(3+1)$ dimensions de Sitter space has a smooth conformal compactification. By use of the conformal method, the study of global stability is reduced to a finite in time problem for the conformal equations, which can be written as a symmetric hyperbolic system. Additionally, this method also gives a scattering theory between asymptotic data at $\mathcal{I}^-$ and asymptotic data at $\mathcal{I}^+$, which represent two regular spacelike hypersurfaces in the conformal spacetime. We note that this result was also generalized to the Einstein-Maxwell-Yang-Mills system in [Fri91] and to the Einstein-radiative fluid system in [LVK13].

In the case of the Einstein equations coupled to a non-linear scalar field, which is a generalization of (1.1), Ringström proved stability in all dimensions in [Rin08] for small Cauchy initial data on a finite time spacelike hypersurface. This proof is robust in order to treat such general equations, but it does not give a description of the induced scattering data at infinity.

1.1.2 The Fefferman-Graham Expansion

The starting point in the theory of local well-posedness with scattering data at $\mathcal{I}^-$ for all $n \geq 3$ is given by the work of Fefferman-Graham [FG85, FG12].

To construct conformal invariants for an n-dimensional Riemannian manifold $\big(\mathcal{S}, \slashed{g}_0\big)$, Fefferman and Graham first consider the corresponding ambient metric. We briefly introduce the ambient metric construction here and we discuss it in detail in Section 1.2. For any $\slashed{g}_0$ and any symmetric traceless 2-tensor $\check{h}$, the ambient metric is an $(n+2)$-dimensional self-similar vacuum metric given by a formal power series expansion determined by $\big(\slashed{g}_0, \check{h}\big)$. The conformal invariants of $\big(\mathcal{S}, \slashed{g}_0\big)$ are then obtained using the classification of local pseudo-Riemannian invariants of the ambient metric. According to [FG12], under the additional assumption of straightness on $\check{h}$, which determines the divergence of $\check{h}$ in terms of $\slashed{g}_0$, the ambient metric is straight and one can take the quotient by the action of the scaling vector field to obtain formal asymptotically de Sitter solutions of (1.1).

We illustrate the Fefferman-Graham expansion of formal asymptotically de Sitter vacuum solutions. In this setting, the formal power series expansions (1.9)-(1.10) were first constructed in [Ren04, Theorems 2, 3]. The scattering data are given by a Riemannian metric $\big(S^n, \slashed{g}_0\big)$ and a symmetric traceless straight 2-tensor $\check{h}$, which determine each term in the expansion. For $n \geq 3$ odd, the expansion at $\mathcal{I}^-$ is smooth in terms of e^T:

$$\widetilde{\slashed{g}} = \slashed{g}_0 + \frac{e^{2T}}{2^2}\slashed{g}_1 + \ldots + \frac{e^{(n-1)T}}{2^{n-1}((n-1)/2)!}\slashed{g}_{(n-1)/2} + \frac{e^{nT}}{2^n}\check{k} + O\big(e^{(n+1)T}\big). \tag{1.9}$$

In the case of $n \geq 4$ even, we have the expansion at $\mathcal{I}^-$:

$$\widetilde{\slashed{g}} = \slashed{g}_0 + \frac{e^{2T}}{2^2}\slashed{g}_1 + \ldots + \frac{e^{(n-2)T}}{2^{n-2}(n/2-1)!}\slashed{g}_{n/2-1} + \frac{2Te^{nT}}{2^n(n/2)!}\mathcal{O} + \frac{e^{nT}}{2^n(n/2)!}\check{k} + O\big(Te^{(n+2)T}\big). \tag{1.10}$$

The compatibility relations are obtained by taking the limit of (1.1) at $\mathcal{I}^-$ at each order. The terms of order less than e^{nT} are determined by $\slashed{g}_0$. We also have that $\mathrm{tr}\check{k}$ is determined by $\slashed{g}_0$, and that the trace-free part of $\check{k}$ is $\check{h}$. Finally, all the higher order terms in the expansion are determined by $\slashed{g}_0$ and $\check{h}$.

1.1.3 The Local Well-posedness Theory with Scattering Data

The expansions (1.9) and (1.10) were computed formally in the smooth category in [FG85, Ren04, FG12], and convergence was only proven in the case of analytic scattering data [Ren04, Theorem 6]. The rigorous proof of the Fefferman-Graham expansion in the smooth case was done in [RSR18] in a more general context (and revisited in [Hin24]). Restricted to our situation, the results of [RSR18] imply the following local well-posedness result with scattering data:

Theorem 1.2 ([RSR18])**.** *For any* $n \geq 3$ *and any smooth straight scattering data* $\left(\not{g}_0, \check{h}\right)$*, there exists a unique solution of (1.1) of the form (1.3) in a neighborhood of* $\mathcal{I}^-$ *which satisfies the expansions* (1.9) *and* (1.10)*.*

The local well-posedness result of [RSR18] is a fundamental ingredient needed to study the long time behavior of asymptotically de Sitter solutions of (1.1). For simplicity, we only stated how the results of [RSR18] apply in our situation of straight ambient metrics. However, the results of [RSR18] hold in the very general context of ”proto-ambient metrics”, which only require the Fefferman-Graham expansion to hold up to the term containing $\check{k}$.

1.1.4 A Scattering Theory in the Odd Spatial Dimension Case

The results of [Fri83, Fri86] were further generalized in [And05] for all $(n+1)$-dimensional de Sitter spaces with $n \geq 3$ odd. While the conformal method does not apply in higher dimensions, there is nevertheless the simplification of having the expansion (1.9) which is smooth in terms of e^T. Moreover, in this case the Einstein equations (1.1) can be replaced by the equation $\mathcal{O} = 0$, which is conformally invariant and leads to a hyperbolic system in a suitable gauge. We notice that in $(3+1)$ dimensions the obstruction tensor $\mathcal{O}$ coincides with the Bach tensor, so the approach of [And05] is to replace (1.1) by the Bach equations. Using these ingredients, [And05] generalizes the conformal method proof to obtain stability for all $n \geq 3$ odd, which also gives a scattering theory between asymptotic data at $\mathcal{I}^-$ and asymptotic data at $\mathcal{I}^+$.

1.1.5 The Wave Equation on de Sitter Space

The simplest model problem needed in order to understand the scattering of asymptotically de Sitter vacuum solutions is the linear wave equation on a fixed de Sitter background:

$$\Box_{dS}\widetilde{\phi} = 0. \tag{1.11}$$

The scattering problem in the more general case of the Klein-Gordon equation was addressed in [Vas10]. Given a certain relation between the Klein-Gordon mass and the spatial dimension, which guarantees a smooth expansion at infinity similar to (1.9), [Vas10] provides a detailed description of the scattering map as a Fourier integral operator. However, in the case of $n \geq 4$ even and vanishing Klein-Gordon mass, the solution satisfies an expansion at infinity similar to (1.10). The results of [Vas10] prove that the scattering map is an isomorphism on C^∞.

In the case of (1.11) with $n \geq 4$ even, we used a different approach in [Cic23] to construct the scattering map as a Banach space isomorphism for asymptotic initial data $\big(\phi_0, \mathfrak{h}\big) \in H^{M+n}(S^n) \times H^M(S^n)$, for any $M \geq 1$. We notice that ϕ_0 plays a similar role to $\not{g}_0$, and $\mathfrak{h}$ is again obtained by renormalizing a higher order term in the Fefferman-Graham expansion using $\log \nabla$ of the analogue of the obstruction tensor. Based on these similarities, [Cic23] will provide the guideline for studying the scattering of asymptotically de Sitter vacuum solutions for all $n \geq 4$ even in the present work. It turns out that the methods used in [Cic23] are indeed robust and can be adapted in the current setting.

1.1.6 Results for the Expanding Region of Black Hole Spacetimes

Similarly to the case of asymptotically de Sitter vacuum spacetimes, it is of great interest to study the question of scattering in the expanding region of black hole spacetimes with $\Lambda > 0$, which remains an open problem even in $(3+1)$ dimensions. We first mention [Ber25], which studies scattering for the linear wave equation on the expanding region of Schwarzschild-de Sitter spacetimes. In the nonlinear case, the recent works [FS24, HV24] prove the stability of the expanding region of Kerr-de Sitter, with [HV24] also establishing conformal smoothness at $\mathcal{I}^+$.

1.1.7 Scattering Results in the Asymptotically Flat Setting

In the asymptotically flat setting, [Wan10] constructs a scattering isomorphism between Cauchy initial data and the radiation field for perturbations of Minkowski space in $(n+1)$-dimensions, with $n \geq 4$. In the case of $n = 3$, [BSB15] proves scattering for certain semilinear wave equations on Minkowski space.

We note that the study of scattering in the context of asymptotically flat black hole spacetimes is a current area of research. We refer the reader to [BW14], [DHR24], [DRSR18], [Alf20], [Mas22], and references therein.

1.2 The Ambient Metric Formulation

The ambient metric construction provides an embedding of solutions of (1.1) of the form (1.3) into $(n+2)$-dimensional self-similar vacuum spacetimes. The simplest example for this correspondence is that of de Sitter space $\big(\mathbb{R} \times S^n, \widetilde{g}_{dS}\big)$. The associated straight ambient metric is the $\{u < 0, v > 0\} \times S^n \subset \mathbb{R}^{n+2}$ region of Minkowski space with the Minkowski metric m, where u and v are the standard double null coordinates.

The embedding allows us to prove a scattering result at the level of the corresponding ambient metric instead. In the $n \geq 4$ even case the solution is not smooth at infinity, so we can interpret this construction as a compactification that allows us to reduce a global problem with data at infinity to a finite problem with singular data. Another advantage of this setting is that the ambient spacetime has a natural double null foliation and we can use the approach developed in [RSR18] and [RSR23]. Moreover, the explicit embedding provides additional structure on the ambient metric, as can be seen in the definition:

Definition 1.1. *Let $I \subset \mathbb{R}$ be an open interval, set $\widetilde{\mathcal{M}} = I \times S^n$, and let $\big(\widetilde{\mathcal{M}}, \widetilde{g}\big)$ be a solution of (1.1) of the form:*

$$\widetilde{g} = -dT^2 + e^{-2T} \widetilde{\not{g}}_{AB}(T, \theta^1, \ldots, \theta^n) d\theta^A d\theta^B. \tag{1.12}$$

We define the corresponding straight ambient metric to be $\big(\mathcal{M}, g\big)$, where $\mathcal{M} = (-\infty, 0) \times \widetilde{\mathcal{M}}$ and:

$$g = ds^2 + s^2\big(- dT^2 + e^{-2T} \widetilde{\not{g}}_{AB}(T, \theta^1, \ldots, \theta^n) d\theta^A d\theta^B\big). \tag{1.13}$$

The spacetime $\big(\mathcal{M}, g\big)$ *is an* $(n{+}2)$*-dimensional straight self-similar vacuum spacetime, satisfying:*

$$Ric(g) = 0,\ \mathcal{L}_S g = 2g,\ S = s\partial_s. \tag{1.14}$$

In general, we say that a metric g *is straight if it takes the form* (1.13)*, i.e. it arises as a cone metric from a metric* $\widetilde{g}$ *of the form* (1.12) *(see [And01]).*

We define the double null coordinates $u < 0,\ v > 0$ *by:*

$$e^T = 2\sqrt{-\frac{v}{u}},\ s = -2\sqrt{-uv}.$$

In double null coordinates, the straight ambient metric has the form:

$$\mathcal{M} = \left\{u < 0,\ v > 0,\ \log 2\sqrt{-\frac{v}{u}} \in I\right\} \times S^n$$

$$g = -2(du \otimes dv + dv \otimes du) + \not{g}_{AB}(u, v, \theta^1, \ldots, \theta^n)d\theta^A d\theta^B, \tag{1.15}$$

where $\not{g}_{AB}(u, v, \theta^1, \ldots, \theta^n) = u^2\widetilde{\not{g}}_{AB}\big(T(u, v), \theta^1, \ldots, \theta^n\big)$*. Moreover, the scaling vector field is* $S = u\partial_u + v\partial_v$*.*

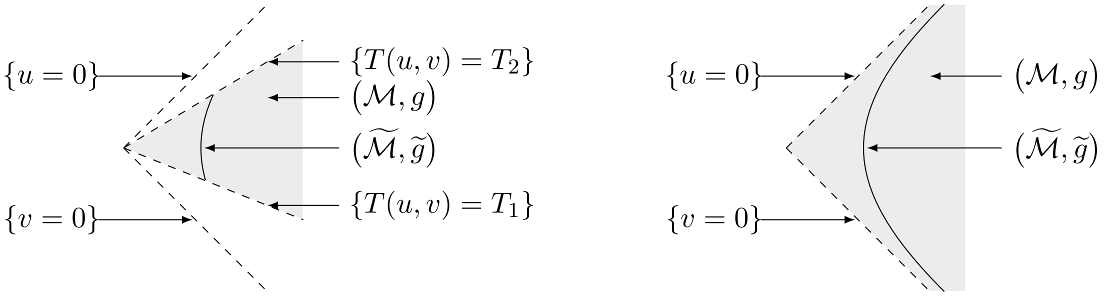

Figure 1.2: Embedding of $\big(\widetilde{\mathcal{M}}, \widetilde{g}\big)$ in the ambient spacetime $\big(\mathcal{M}, g\big)$ in the cases $I = (T_1, T_2)$ and $I = \mathbb{R}$.

In general, we refer to any spacetime $\big(\mathcal{M}, g\big)$ *satisfying the above properties as a straight self-similar vacuum spacetime in double null coordinates. Given any such spacetime, we can quotient by the scaling vector field in order to obtain the corresponding* $(n + 1)$*-dimensional spacetime* $\big(\widetilde{\mathcal{M}}, \widetilde{g}\big)$ *of the form (1.3) which solves (1.1).*

The main point of the ambient metric construction is that establishing a complete scattering theory for solutions of (1.1) with scattering data close to the data of de Sitter space is equivalent to proving a scattering theory for straight self-similar vacuum spacetimes with scattering data on $\{v = 0\}$ or $\{u = 0\}$ close to the data of Minkowski space. This follows since we can identify the

$\mathcal{I}^-$ of $\big(\widetilde{\mathcal{M}},\widetilde{g}\big)$ with the quotient of $\{v=0\}$ by the action of the scaling vector field, and similarly we can identify $\mathcal{I}^+$ with the quotient of $\{u=0\}$ by S. We refer the reader to [And01] and [RSR18] for a general proof of the correspondence.

Notation convention. We use the tilde superscript notation in the original $(n+1)$-dimensional formulation (for example $\widetilde{\mathcal{M}},\widetilde{g},\widetilde{\not{g}}$), and we drop the tilde superscript in the ambient metric formulation (for example $\mathcal{M},g,\not{g}$).

The local well-posedness theory. Before stating our main result in the ambient metric formulation, we outline the local well-posedness theory in the current situation. By self-similarity, to obtain scattering data on $\{v=0\}$ it suffices to specify data on the sphere $\{u=-1,v=0\}\times S^n$. Thus, the notion of scattering data in the ambient metric formulation is the same as in the original $(n+1)$-dimensional formulation. The results of [RSR18] imply that for any smooth straight scattering data $\big(\not{g}_0,\check{h}\big)$ at $\{u=-1,v=0\}\times S^n$, there exists a unique straight self-similar vacuum spacetime in double null coordinates defined in a neighborhood of $\{v=0\}$:

$$\left\{u<0,\ v\geq 0,\ 0\leq -\frac{v}{u}<\underline{v}\right\}\times S^n$$

for some small $\underline{v}>0$ depending on the size of the initial data. Moreover, the solution satisfies the expansion:

$$\begin{aligned}u^{-2}\not{g} =&\not{g}_0+v/|u|\not{g}_1+\ldots+\frac{(v/|u|)^{\frac{n-2}{2}}}{(n/2-1)!}\not{g}_{n/2-1}+\frac{(v/|u|)^{\frac{n}{2}}\log\big(4v/|u|\big)}{(n/2)!}\mathcal{O}\\&+\frac{(v/|u|)^{\frac{n}{2}}}{(n/2)!}\check{k}+O\Big((v/|u|)^{\frac{n+2}{2}}\log\big(v/|u|\big)\Big)\end{aligned}$$

for the same 2-tensors $\not{g}_1,\ldots,\not{g}_{n/2-1},\mathcal{O},\check{k}$ as in (1.4), determined by $\not{g}_0$ and $\check{h}$. The same result holds for data at $\{u=0\}$, upon replacing (u,v) by $(-v,-u)$.

The main result in the ambient metric formulation. We briefly explain the corresponding notion of an asymptotic initial data set in the current setting, and refer the reader to Definition 9.1 for the precise definition.

In what follows, we assume some familiarity with the double null formalism introduced in detail in Section 2. We denote the Ricci coefficients schematically by ψ and the curvature components by Ψ. We consider the case of scattering data at $\{u=-1,v=0\}\times S^n$, and note that the case of scattering data at $\{u=0,v=1\}\times S^n$ is defined similarly by replacing (u,v) with

$(-v,-u)$. Given smooth scattering data $\big(\slashed{g}_0,\check{h}\big)$, we define the *asymptotic initial data set* $\Sigma\big(\slashed{g}_0,\check{h}\big)$ to be the collection of tensors on $\{u=-1,v=0\}\times S^n$ consisting of: the metric $\slashed{g}_0$; the double null quantities $\slashed{\psi}$ and Ψ, together with certain angular and ∇_{∂_v} derivatives of these tensors, which can be computed by the compatibility relations in terms of $\slashed{g}_0$ (as in [RSR18], the specification of these tensors is equivalent to the specification of $\slashed{g}_1,\dots,\slashed{g}_{n/2-1},\mathrm{tr}\check{k}$); the obstruction tensor $\mathcal{O}$; and the renormalized tensor $\check{\mathfrak{h}}=\check{h}-2\big(\log\nabla\big)\mathcal{O}$. Next, we define the *asymptotic initial data norm*, measuring closeness to the Minkowski data. For any tensor $\phi\in\Sigma\big(\slashed{g}_0,\check{h}\big)$, we denote by $\phi^*=\phi-\phi_{\text{Minkowski}}$ the tensor obtained as the difference of ϕ and its Minkowski value. As before, we define the *asymptotic initial data norm of order M* by:

$$\Big\|\Sigma\big(\slashed{g}_0,\check{h}\big)\Big\|_M^2=\sum_{\phi\in\Sigma(\slashed{g}_0,\check{h})}\|\phi^*\|^2_{H^{M+1}(S^n)}.$$

We denote by $\Sigma_{\text{Minkowski}}=\Sigma\big(\frac{1}{4}\slashed{g}_{S^n},0\big)$ the initial data set corresponding to Minkowksi space. For any $\epsilon>0$, we define the set of smooth ϵ-small asymptotic data of order M by:

$$B^M_\epsilon\big(\Sigma_{\text{Minkowski}}\big)=\Big\{\Sigma\big(\slashed{g}_0,\check{h}\big):\ \big\|\Sigma\big(\slashed{g}_0,\check{h}\big)\big\|_M<\epsilon\Big\}.$$

Remark 1.3. *In order to make the previous definition of $\widetilde{\Sigma}\big(\slashed{g}_0,\check{h}\big)$ precise, we require that the norms $\big\|\Sigma\big(\slashed{g}_0,\check{h}\big)\big\|_M$ and $\big\|\widetilde{\Sigma}\big(\slashed{g}_0,\check{h}\big)\big\|_M$ are equivalent, where $\Sigma\big(\slashed{g}_0,\check{h}\big)$ is given as in Definition 9.1. This determines the exact components that are contained in the set $\widetilde{\Sigma}\big(\slashed{g}_0,\check{h}\big)$.*

Using the ambient metric construction, we can restate Theorem 1.1 in the following equivalent formulation:

Theorem 1.3. *For any even integer $n\geq 4$, we have a complete scattering theory for $(n+2)$-dimensional straight self-similar vacuum spacetimes determined by small data. For any $M>0$ large enough there exists $\epsilon_0>0$ small enough, such that for $0<\epsilon\leq\epsilon_0$ we have:*

1. ***Existence and uniqueness of scattering states:*** *for any smooth ϵ-small asymptotic data of order M at $\{v=0\}$ given by $\Sigma\big(\slashed{g}_0,\check{h}\big)\in B^M_\epsilon\big(\Sigma_{\text{Minkowski}}\big)$, there exists a unique smooth straight self-similar vacuum solution $\big(\mathcal{M},g\big)$ in double null coordinates defined globally in $\{u<0,\ v>0\}\times S^n$, which remains quantitatively close to Minkowski space, in the sense of Propositions 3.1, 3.2, and 3.8. Moreover, the solution extends to $(-\infty,0)\times[0,\infty)\times S^n$ as a regular solution of the Einstein vacuum equations (1.14) in the sense of Definition 2.3;*

2. ***Asymptotic completeness:*** *any smooth straight self-similar vacuum spacetime in double null coordinates which is quantitatively close in the sense of Remark 3.8 to Minkowski space on a spacelike hypersurface* $\{v = c|u|\}$, *can be extended to the region* $\{u < 0,\ v > 0\} \times S^n$ *and induces smooth scattering data* $\big(\slashed{g}_0, \check{h}\big)$ *at* $\{v = 0\}$ *and* $\big(\underline{\slashed{g}_0}, \underline{\check{h}}\big)$ *at* $\{u = 0\}$*. Moreover, the solution extends to* $\big((-\infty, 0] \times [0, \infty) \backslash \{(0, 0)\}\big) \times S^n$ *as a regular solution of the Einstein vacuum equations* (1.14) *in the sense of Definition 2.3;*

3. ***Existence of a scattering map with quantitative estimates:*** *there exists a constant* $C_M > 0$ *independent of* ϵ, *such that we have a well-defined scattering map taking the asymptotic data at* $\{v = 0\}$ *to asymptotic data at* $\{u = 0\}$*:*
$$\begin{cases} \mathcal{S} : B^M_\epsilon\big(\Sigma_{\text{Minkowski}}\big) \to B^M_{C_M\epsilon}\big(\Sigma_{\text{Minkowski}}\big), \\ \mathcal{S}\Big(\Sigma\big(\slashed{g}_0, \check{h}\big)\Big) = \Sigma\big(\underline{\slashed{g}_0}, \underline{\check{h}}\big). \end{cases} \tag{1.16}$$
The scattering map is locally invertible and locally Lipschitz at $\Sigma_{\text{Minkowski}}$, *in the sense that it satisfies the quantitative estimates:*
$$\Big\|\mathcal{S}\big(\Sigma(\slashed{g}_0, \check{h})\big)\Big\|_M \leq C_M \Big\|\Sigma\big(\slashed{g}_0, \check{h}\big)\Big\|_M. \tag{1.17}$$

Remark 1.4. *As in Theorem 1.1, the scattering map result is sharp and avoids any "derivative loss", since in* (1.16) *and* (1.17) *we use the same Sobolev-type norm to measure the smallness of asymptotic data at* $\mathcal{I}^-$ *and* $\mathcal{I}^+$*.*

Remark 1.5. *The ambient metric formulation is convenient in order to establish the existence, uniqueness of scattering states, and asymptotic completeness. However, one could work directly at the level of asymptotically de Sitter spaces of the form* (1.3)*. In this case it is convenient to use the time coordinate* $\tau = e^T/2$*. This approach is present in Section 7, Section 8, and [Cic26].*

1.3 Outline of the Proof

We present the main steps in the proof of Theorem 1.3. We assume familiarity with the basics of the double null formalism, and we refer the reader unfamiliar with these notions to read Section 2 for a detailed introduction.

We recall that in this formalism, the Einstein vacuum equations (1.14) can be written as a system of equations for the Ricci coefficients denoted by ψ and the curvature components denoted by Ψ. The system has the following schematic form (see [RSR18]):

$$\begin{cases} \nabla_3\psi = \Psi + \psi\cdot\psi,\ \nabla_4\psi = \Psi + \psi\cdot\psi, \\ \nabla_3\Psi_1 = \mathcal{D}\Psi_2 + \psi\cdot\Psi,\ \nabla_4\Psi_2 = -\mathcal{D}^*\Psi_1 + \psi\cdot\Psi, \end{cases} \tag{1.18}$$

where $\mathcal{D}, \mathcal{D}^*$ are adjoint differential operators on S^n, and ∇_3, ∇_4 are covariant derivatives in the $e_3 = \partial_u, e_4 = \partial_v$ directions. We denote by ∇ the projection to the tangent space of S^n of the covariant derivative in any direction tangent to S^n, and we refer to this differential operator as an angular derivative. The operators ∇_3, ∇_4, and ∇ will be used as commutators to obtain systems of equations with a similar form to (1.18). We also point out that the system (1.18) has significant simplifications, due to the special straight structure of the metric g which has constant lapse and vanishing shift vector.

1.3.1 Existence and Uniqueness of Scattering States

The first statement of Theorem 1.3 consists of global existence and quantitative estimates of the solution in the $(n+2)$-dimensional region $\{u<0,\ v>0\}\times S^n$, given small scattering data at $\{v=0\}$. We prove this in Theorems 3.1 and 3.2. In the original $(n+1)$-dimensional formulation, this result represents the global stability of de Sitter space with small scattering data at $\mathcal{I}^-$. We point out that the proof of [Rin08] does not apply in the case of scattering data; additionally, we prefer to prove the needed stability result in the ambient metric setting, in order to obtain the estimates required for the rest of our proof.

We remark that the stability result that we prove at this stage is not optimal in terms of the smallness assumed on the initial data. For our purposes, we notice that $\Sigma\big(\not{g}_0, \check{h}\big) \in B^M_\epsilon\big(\Sigma_{\text{Minkowski}}\big)$ implies:

$$\big\|\not{g}_0^*\big\|_{\mathring{H}^M(S^n)} + \big\|\mathcal{O}\big\|_{H^M(S^n)} + \big\|\check{h}\big\|_{H^M(S^n)} \leq \epsilon, \tag{1.19}$$

where $\mathring{H}^M(S^n)$ is the Sobolev space with respect to $\not{g}_{S^n}$ and $H^M(S^n)$ is the Sobolev space with respect to $\not{g}_0$. For this part of the argument we use the smallness condition (1.19) instead of $\Sigma\big(\not{g}_0, \check{h}\big) \in B^M_\epsilon\big(\Sigma_{\text{Minkowski}}\big)$. However, we point out that in order to prove the sharp estimate for the scattering map (1.17), we will need a more detailed analysis which makes use of the exact structure of $\Sigma\big(\not{g}_0, \check{h}\big)$.

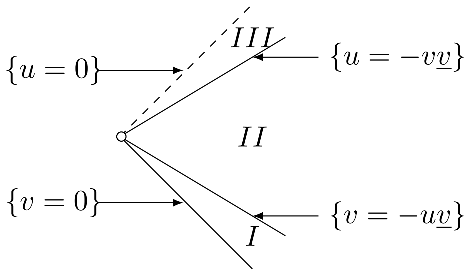

Figure 1.3: The decomposition into the regions I, II, and III

The strategy of the proof follows similar steps to [RSR23]. In Section 3, we carry out a bootstrap argument and construct the solution in the following regions one at a time, for $\underline{v} > 0$ sufficiently small:

$$I = \left\{0 \leq \frac{v}{|u|} \leq \underline{v}\right\} \times S^n,\ II = \left\{\underline{v} \leq \frac{v}{|u|} \leq \underline{v}^{-1}\right\} \times S^n,\ III = \left\{0 < \frac{|u|}{v} \leq \underline{v}\right\} \times S^n$$

The bounds that we prove for the solution need to be consistent with self-similarity, as in [RSR18]. In region I we have the scaling for the Ricci coefficients $|\psi^*| \sim \epsilon|u|^{-1}$ and for the curvature components $|\Psi| \sim \epsilon|u|^{-2}$. Moreover, each $\nabla,\ \nabla_3$, or ∇_4 derivative that we apply to the double null unknowns raises their homogeneity by one, implying schematic self-similar bounds of the form:

$$\left|\nabla^i\nabla_4^j\nabla_3^k\psi^*\right| \lesssim_{i,j,k} \epsilon|u|^{-1-i-j-k},\ \left|\nabla^i\nabla_4^j\nabla_3^k\Psi\right| \lesssim_{i,j,k} \epsilon|u|^{-2-i-j-k}. \tag{1.20}$$

The expected bounds in regions II and III are similar for most double null quantities, replacing $|u|$ with v.

An essential aspect of the problem is that we can take at most $\frac{n-4}{2}$ ∇_4 derivatives of the double null quantities in a neighborhood of $\{v=0\}$, and similarly for ∇_3 at $\{u=0\}$. This results from the presence of the obstruction term $\mathcal{O}$ in the Fefferman-Graham expansion of the solution near $\{v=0\}$. We denote by Ψ^G any curvature component excluding α, and similarly by $\underline{\Psi}^G$ any curvature component excluding $\underline{\alpha}$. We have that all double null quantities $\nabla^i\nabla_4^j\nabla_3^k\psi$ and $\nabla^i\nabla_4^j\nabla_3^k\Psi^G$ extend to $\{v=0\}$ for $j \leq \frac{n-4}{2}$. However, α is mildly singular at $\{v=0\}$:

$$|u|^{\frac{n-4}{2}}\nabla_4^{\frac{n-4}{2}}\alpha = \log\big(v/|u|\big)\mathcal{O} + h + O\Big(v/|u|\log\big(v/|u|\big)\Big), \tag{1.21}$$

where h is obtained from $\check{h}$ by subtracting a linear factor of $\mathcal{O}$. In order to address this issue, when proving self-similar bounds for α in region I we need to subtract off the singular term in (1.21).

Taking into consideration the necessary renormalization, the argument of [RSR18] applies to prove existence and self-similar bounds in region I, where we allow up to N angular derivatives on the double null quantities, and $M = N + O(n),\ M > N$. Similarly, the argument of [RSR23] applies to prove existence and self-similar bounds in region II, again allowing up to N angular derivatives on the double null quantities.

The main part of the proof of the first statement in Theorem 1.3 involves showing existence and self-similar bounds in region III in Section 3.2. The difficult aspect is that we expect $\underline{\alpha}$ to be singular at $\{u = 0\}$, as implied by the local well-posedness theory. Unlike in region I, we do not determine a priori the singular part of $\underline{\alpha}$ given by the obstruction tensor, so we cannot subtract it off. We also notice that unlike the approach in [RSR23], we cannot work with the reduced Bianchi system, which would remove $\underline{\alpha}$, as this does not work in the higher dimensional setting. Our solution is to propagate estimates for $\underline{\alpha}$ consistent with it blowing up at $\{u = 0\}$, aided by the fact that in the straight higher dimensional case the singular behavior of $\underline{\alpha}$ is more mild than in [RSR23]. For the remaining double null quantities, we expect to prove regular self-similar bounds, similar to region I.

We briefly explain how to prove energy estimates for the curvature components Ψ, as part of the bootstrap argument in Section 3.2. We recall that the curvature components can be grouped into the Bianchi pairs $(\alpha, \nu), (\nu, R), (R, \underline{\nu}), (\underline{\nu}, \underline{\alpha})$. The Bianchi pair $(\underline{\nu}, \underline{\alpha})$ satisfies the schematic equations:

$$\nabla_3 \underline{\nu}_{ABC} = -2\nabla_{[A}\underline{\alpha}_{B]C} + \dots$$

$$\nabla_4 \underline{\alpha}_{AB} + \frac{n}{2v}\underline{\alpha}_{AB} = -\nabla^C \underline{\nu}_{C(AB)} + \dots$$

As in [RSR23], for $0 < q \ll p \ll 1$ we conjugate the equations with $w = v^{\frac{3}{2}-p}|u|^{p-q}$:

$$\nabla_3 w\underline{\nu}_{ABC} + \frac{p-q}{|u|} w\underline{\nu}_{ABC} = -2\nabla_{[A} w\underline{\alpha}_{B]C} + \dots$$

$$\nabla_4 w\underline{\alpha}_{AB} + \left(\frac{n-3}{2} + p\right)\frac{w}{v}\underline{\alpha}_{AB} = -\nabla^C w\underline{\nu}_{C(AB)} + \dots$$

The energy estimates are obtained by contracting these equations with $w\underline{\nu}$ and $w\underline{\alpha}$, integrating by parts, and multiplying by $|u|^{2q}$. The lower order terms imply the presence of bulk terms with favorable sign in the estimates. These bulk terms are even better in the case of $\underline{\nu}$, since $|u|/v$ is small in region III. We notice that the same argument also applies when commuting with angular derivatives and up to $\frac{n-4}{2}$ ∇_3 derivatives, which ensures that the lower order terms imply good bulk terms. Commuting with a high number of angular derivatives ∇^i simplifies our treatment of the error terms on the right hand side. The weight w implies that the best estimate that we can prove for $\underline{\alpha}$ is:

$$\left\|\nabla^i\nabla_3^j\underline{\alpha}\right\|_{L^2(S^n)} \lesssim \epsilon^{1-2\delta}|u|^{-p}\cdot|v|^{-2-i-j+p},$$

where $j \leq \frac{n-4}{2}$, $i < N$, $\delta > 0$ is a small constant, and the implicit constant in the inequality is independent of ϵ and $\underline{v}$. This bound is consistent with the singular behavior of α at $\{u = 0\}$. The good bulk term obtained for $\underline{\nu}$ allows us to control the bulk term in the energy estimates for the Bianchi pair $\left(R, \underline{\nu}\right)$, and we similarly obtain energy estimates for all the curvature components. Moreover, the stronger control that we obtain for the bulk terms in the case of the quantities $\underline{\Psi}^G$ allows us to prove regular self-similar bounds. The simple transport structure of the equations for the Ricci coefficients also implies their respective self-similar bounds. We point out that we close the bootstrap assumptions using the smallness of $\underline{v}$.

Finally, we prove a standard propagation of regularity result in Theorem 3.2 showing that if the scattering data is also smooth, the global solution obtained is smooth.

1.3.2 Asymptotic Completeness

We consider a smooth straight self-similar vacuum spacetime in double null coordinates, which is quantitatively close to Minkowski space on a spacelike hypersurface $\{v = c|u|\} \times S^n$ with $\underline{v} \leq c \leq \underline{v}^{-1}$, in the sense that it satisfies the self-similar bounds (1.20) with $j, k \leq \frac{n-4}{2}$ and $i \leq N$, where $M = N + O(n)$, $M > N$ for some large enough N. In Theorem 4.1 we prove the second statement of Theorem 1.3, showing that the spacetime can be extended globally to $\{u < 0, v > 0\} \times S^n$, and that it induces scattering data at $\{v = 0\}$ and $\{u = 0\}$. The first part follows from Section 1.3.1, as we remark that the analysis of [RSR23] in region II applies in this setting as well, and we can repeat our analysis in region III. We notice that in region I the spacetime will satisfy similar bounds to the ones in region III, as we have no information about the scattering data at this point.

In Section 4 we prove the existence of induced smooth scattering data $\big(\underline{\slashed{g}_0}, \underline{\check{h}}\big)$ at $\{u=0\}$, as the case of $\{v=0\}$ is analogous. The strategy is to compute the terms in the expansion of $\slashed{g}$ at $\{u=0\}$ up to order $n/2$.

We first prove that certain regular quantities can be extended to $\{u=0\}$ and satisfy compatibility relations. These consist of up to $\frac{n-6}{2}$ ∇_3 derivatives of $\underline{\alpha}$, up to $\frac{n-4}{2}$ ∇_3 derivatives of $\underline{\Psi}^G$, up to $\frac{n-4}{2}$ ∇_3 derivatives of ψ, and up to $\frac{n-2}{2}$ $\mathcal{L}_3$ derivatives of $\slashed{g}$, together with at most N angular derivatives of these tensors. All these quantities satisfy a ∇_3 equation, where we control the right hand side using the bounds proved in region III. We prove that these tensors are in $W^{1,1}_u\big([-1,0]\big)L^2(S^n)$, so they can be extended to $\{u=0\}$. In particular, we compute $\underline{\slashed{g}_0}$ the induced metric on $\{u=0, v=1\}\times S^n$. Evaluating these ∇_3 equations at $\{u=0\}$ implies that the regular quantities are determined in terms of $\underline{\slashed{g}_0}$ by the compatibility relations of [RSR18]. Equivalently, we obtain that the first $\frac{n-2}{2}$ terms in the expansion of $\slashed{g}$ are determined by $\underline{\slashed{g}_0}$ via the compatibility relations.

The next step is to compute the singular component of $\nabla_3^{\frac{n-4}{2}}\underline{\alpha}$ and to prove that it is given by the obstruction tensor of $\underline{\slashed{g}_0}$. Using self-similarity, we can write the ∇_4 Bianchi equation for $\nabla_3^{\frac{n-4}{2}}\underline{\alpha}$ schematically as:

$$\partial_u\big(v^{\frac{n-4}{2}}\nabla_3^{\frac{n-4}{2}}\underline{\alpha}\big) = -\frac{1}{u}\mathcal{E}_1 + \mathcal{E}_2 = \frac{1}{u}\underline{\mathcal{O}} + \frac{1}{|u|}\Big(\mathcal{E}_1 - \mathcal{E}_1|_{u=0}\Big) + \mathcal{E}_2, \tag{1.22}$$

where we defined $\underline{\mathcal{O}} = -\mathcal{E}_1|_{u=0}$ which is independent of v and can be computed in terms of the regular quantities at $\{u=0\}$. We obtain that $\underline{\mathcal{O}}$ can be computed in terms of $\underline{\slashed{g}_0}$, and we prove it satisfies the compatibility relation of [RSR18] which implies that it represents the obstruction tensor of $\underline{\slashed{g}_0}$.

The final step is to compute the induced tensor $\underline{\check{h}}$ on $\{u=0\}$. We notice that the error term in (1.22) is in $L^1_u\big([-\underline{v}v, 0]\big)L^2(S^n)$, so we can integrate the equation to get:

$$v^{\frac{n-4}{2}}\nabla_3^{\frac{n-4}{2}}\underline{\alpha} - \underline{\mathcal{O}}\log\big(|u|/v\big) \in W^{1,1}_u\big([-\underline{v}v, 0]\big)L^2(S^n). \tag{1.23}$$

We define the symmetric traceless 2-tensor $\underline{h}$ which is independent of u and v to be the limit at $\{u=0\}$ of (1.23). We obtain the expansion for $\nabla_3^{\frac{n-4}{2}}\underline{\alpha}:$

$$v^{\frac{n-4}{2}}\nabla_3^{\frac{n-4}{2}}\underline{\alpha}=\log\big(|u|/v\big)\underline{\mathcal{O}}+\underline{h}+O\big(|u|^{1-p}/v^{1-p}\big).$$

As before, $\underline{\check{h}}$ is obtained from $\underline{h}$ by adding a certain linear factor of $\underline{\mathcal{O}}$.

The proof of asymptotic completeness is concluded by applying Theorem 1.2 of [RSR18], in order to show that $\big(\underline{\not{g}_0},\underline{\check{h}}\big)$ represents the induced scattering data at $\{u=0\}$. This result also implies that $\underline{h}$ satisfies the straightness condition, since the spacetime $\big(\mathcal{M},g\big)$ is straight. Finally, we remark that the spacetime $\big(\mathcal{M},g\big)$ is smooth, but due to the mild singular behavior at $\{u=0\}$ and $\{v=0\}$ it extends to $\big((-\infty,0]\times[0,\infty)\backslash\{(0,0)\}\big)\times S^n$ as a regular solution of the Einstein vacuum equations (1.14) in the sense of Definition 2.3, similarly to the solutions of [RSR18].

1.3.3 The Scattering Map

The third statement of Theorem 1.3 consists of constructing the scattering map $\mathcal{S}$ according to (1.16), which satisfies the sharp estimate (1.17). Establishing the sharp result for the scattering map represents the most challenging part of our work. The reader might wish to return to this section for assistance while reading the proof in Sections 5-11.

We first explain the preliminary scattering result obtained from the proofs of existence, uniqueness of scattering states, and asymptotic completeness in Theorems 3.1 and 4.1. For smooth scattering data at $\{v=0\}$ which satisfies the smallness condition:

$$\big\|\not{g}_0^*\big\|_{\mathring{H}^M(S^n)}+\big\|\mathcal{O}\big\|_{H^M(S^n)}+\big\|\check{h}\big\|_{H^M(S^n)}\leq\epsilon,$$

we obtain a smooth straight self-similar vacuum spacetime $\big(\mathcal{M},g\big)$ in double null coordinates defined in the region $\{u<0,\ v>0\}\times S^n$. This induces smooth scattering data at $\{u=0\}$ satisfying the smallness condition:

$$\big\|\underline{\not{g}_0}^*\big\|_{\mathring{H}^N(S^n)}+\big\|\underline{\mathcal{O}}\big\|_{H^N(S^n)}+\big\|\underline{\check{h}}\big\|_{H^N(S^n)}\leq C\epsilon^{1-2\delta},$$

where $M=N+O(n),\ M>N$ for large enough N, $\delta>0$ is a small constant, and $C>0$ is a constant independent of ϵ. This confirms our previous claim in Section 1.3.1 that the stability result proved initially is not optimal in terms of the smallness assumptions on the initial data (1.19). In particular, the above result cannot give sharp estimates for the scattering map at this stage, since we only get control of the H^N norm of the solution at $\{u = 0\}$, despite starting with bounds on the H^M norm of the solution at $\{v = 0\}$, with $M > N$. This issue is a fundamental feature of the problem, already present at the level of the wave equation (1.11) which was analyzed in [Cic23].

In order to prove a sharp scattering result, we must construct a notion of an asymptotic initial data set $\Sigma\big(\slashed{g}_0, \check{h}\big)$ and asymptotic initial data norm $\|\cdot\|_M$, which in the small data case allow us to prove the estimate:

$$\Big\|\Sigma\big(\underline{\slashed{g}_0}, \underline{\check{h}}\big)\Big\|_M \leq C_M \Big\|\Sigma\big(\slashed{g}_0, \check{h}\big)\Big\|_M. \tag{1.24}$$

This requires a detailed analysis of the problem which exploits the structure of the solution, and ultimately relies on replacing h with the renormalized tensor $\mathfrak{h} = h - 2\big(\log \nabla\big)\mathcal{O}$.

Once we prove that for any $\Sigma\big(\slashed{g}_0, \check{h}\big) \in B^M_\epsilon\big(\Sigma_{\text{Minkowski}}\big)$ the estimate (1.24) holds, it is straightforward in Section 11 to construct the scattering map $\mathcal{S}$ satisfying (1.16) and (1.17), by also using the existence, uniqueness of scattering states, and asymptotic completeness. We outline the proof of (1.24) for the rest of the section.

We introduce the norm Ξ_M in Section 9, representing the energy of the solution on $\{u = -1, v = 1\} \times S^n$. One remarkable aspect is that the norm Ξ_M has improved angular control on the solution compared to the asymptotic data norm, by gaining half of a derivative. We have schematically that:

$$\Big\|\Sigma\big(\slashed{g}_0, \check{h}\big)\Big\|^2_M = \|\mathcal{O}\|^2_{H^{M+1}(S_{-1,0})} + \|\mathfrak{h}\|^2_{H^{M+1}(S_{-1,0})} + \big\|\nabla_4^{\frac{n-4}{2}}\Psi^G\big\|^2_{H^{M+1}(S_{-1,0})} + \ldots$$

$$\Xi^2_M = \sum_{i+j=0}^{\frac{n-4}{2}} \big\|\nabla^M \nabla_3^i \nabla_4^j \Psi\big\|^2_{H^{3/2}(S_{-1,1})} + \sum_{i+j=0}^{\frac{n-2}{2}} \big\|\nabla^M \nabla_3^i \nabla_4^j \Psi\big\|^2_{H^{1/2}(S_{-1,1})} + \ldots$$

In order to prove (1.24), it suffices to show that:

$$\Big\|\Sigma\big(\slashed{g}_0, \check{h}\big)\Big\|_M \lesssim \Xi_M \lesssim \Big\|\Sigma\big(\slashed{g}_0, \check{h}\big)\Big\|_M, \tag{1.25}$$

where the implicit constant depends on M but is independent on ϵ. Once we establish (1.25), we complete the proof of (1.24) in Section 11, since by changing (u, v) to $(-v, -u)$ we also obtain:

$$\Big\|\Sigma\big(\underline{\slashed{g}_0}, \underline{\check{h}}\big)\Big\|_M \lesssim \Xi_M \lesssim \Big\|\Sigma\big(\underline{\slashed{g}_0}, \underline{\check{h}}\big)\Big\|_M.$$

Remark 1.6. *The strategy used in the proof of the sharp scattering result is similar to our approach in [Cic23] for the wave equation (1.11). Based on the analogy between the scalar field*

ϕ and $\slashed{g}$, one could expect that a notion of asymptotic initial data could consist only of $\slashed{g}_0$ and $\mathfrak{h}$ which satisfy the smallness assumption:

$$\|\slashed{g}_0^*\|_{\mathring{H}^{M+n+1}(S^n)}+\|\mathfrak{h}\|_{H^{M+1}(S^n)}\leq\epsilon. \tag{1.26}$$

Using the compatibility relations, this condition implies that $\Sigma\big(\slashed{g}_0,\check{h}\big)\in B^M_{C_M\epsilon}\big(\Sigma_{\text{Minkowski}}\big)$. In the case of (1.11) we also have that $\mathcal{O}\sim\Delta^{n/2}\phi_0+\dots$, which recovers the estimate for ϕ_0 at top order. However, in the current situation the obstruction tensor does not satisfy the needed ellipticity property, so the smallness of $\Sigma\big(\slashed{g}_0,\check{h}\big)$ does not imply (1.26). According to [FG85], we have at top order that $\mathcal{O}\sim\Delta^{n/2-2}B$, where B is the Bach tensor of $\slashed{g}_0$. This operator is elliptic under a conformal change of the metric, see [TV05, LS16], but in our case we cannot control the conformal factor. Consequently, the smallness condition (1.26) cannot be used to prove a sharp scattering result.

Estimates from $\{v=0\}$ to $\{v=-u\}$. We prove that $\Xi_M\lesssim\big\|\Sigma(\slashed{g}_0,\check{h})\big\|_M$ in Theorem 9.1, establishing the first inequality in (1.25). According to Section 5, we can rewrite the system of Bianchi equations restricted to $\{u=-1\}$ as a system of wave equations for any $0\leq m\leq M$, $0\leq l\leq\frac{n}{2}-2$:

$$\begin{cases}v\nabla_4^2\nabla^m\nabla_4^l\alpha+\Big(3+l-\frac{n}{2}\Big)\nabla_4\nabla^m\nabla_4^l\alpha-\Delta\nabla^m\nabla_4^l\alpha=\psi\nabla^{m+1}\nabla_4^l\Psi+Err_{ml}^{\Psi}\\ v\nabla_4^2\nabla^m\nabla_4^l\Psi^G+\Big(3+l-\frac{n}{2}\Big)\nabla_4\nabla^m\nabla_4^l\Psi^G-\Delta\nabla^m\nabla_4^l\Psi^G=\psi\nabla^{m+1}\nabla_4^l\Psi+Err_{ml}^{\Psi},\end{cases} \tag{1.27}$$

where the error terms Err_{ml}^{Ψ} are defined in detail in Proposition 5.1. Moreover, the solutions satisfy the expansions at $\{v=0\}$:

$$\begin{cases}\nabla_4^l\Psi^G=\big(\nabla_4^l\Psi^G\big)\big|_{(-1,0)}+O(v),\ \nabla_4^l\alpha=\big(\nabla_4^l\alpha\big)\big|_{(-1,0)}+O\big(v|\log v|^2\big)\ \text{for } l\leq\frac{n-6}{2},\\ \nabla_4^{\frac{n-4}{2}}\Psi^G=\big(\nabla_4^{\frac{n-4}{2}}\Psi^G\big)\big|_{(-1,0)}+O\big(v|\log v|^2\big),\ \nabla_4^{\frac{n-4}{2}}\alpha=\mathcal{O}\log v+h+O\big(v|\log v|^2\big).\end{cases} \tag{1.28}$$

We prove the main estimates for the system (1.27) in Theorem 9.1. The desired inequality will follow, since the initial data energy is controlled by $\big\|\Sigma(\slashed{g}_0,\check{h})\big\|_M$, whereas the energy at $(-1,1)$ controls Ξ_M. The top order estimates require control of the quantity:

$$\mathcal{T}=v^2\big\|\nabla_4\nabla^M\nabla_4^{\frac{n-4}{2}}\Psi\big\|^2_{H^{1/2}(S_{-1,v})}+v\big\|\nabla^M\nabla_4^{\frac{n-4}{2}}\Psi\big\|^2_{H^{3/2}(S_{-1,v})}+\big\|\nabla_4^{\frac{n-4}{2}}\Psi^G\big\|^2_{H^{M+1}(S_{-1,v})}.$$

Bounding this energy represents the fundamental part of the proof, as this captures the need to renormalize h and it implies the improvement in the number of angular derivatives controlled.

To prove this in Section 9, we treat the system (1.27) as a linear system on the background obtained by restricting the metric g to the null cone $\{u = -1\}$, with a general inhomogeneous term in place of Err^{Ψ}_{ml}. We refer to the system (1.33) as the **first model system**, introduced in Section 5, and we explain below how we prove estimates for it in Section 7. Once we bound the top order quantity $\mathcal{T}$ in terms of the initial data energy and the error terms, the remaining bounds follow using more standard energy estimates for the system (1.27). As before, the presence of the nonlinear error terms Err^{Ψ}_{ml} does not create significant difficulties since we commuted with a high number of angular derivatives, so these terms are essentially linear.

Estimates from $\{v = -u\}$ to $\{v = 0\}$. We prove that $\big\|\Sigma(\slashed{g}_0, \check{h})\big\|_M \lesssim \Xi_M$ in Theorem 10.1, establishing the second inequality in (1.25). According to Section 5, we can also rewrite the system of Bianchi equations restricted to $\{u = -1\}$ for any $0 \leq m \leq M$, $0 \leq l \leq \frac{n}{2} - 2$ as:

$$\begin{cases} v\nabla_4^2\nabla^m\nabla_4^l\alpha + \Big(3 + l - \frac{n}{2}\Big)\nabla_4\nabla^m\nabla_4^l\alpha - \Delta\nabla^m\nabla_4^l\alpha = \psi\nabla^{m+1}\nabla_4^l\Psi + Err^{\Psi}_{ml} \\ v\nabla_4^2\nabla^m\nabla_4^l\Psi^G + \Big(2 + l - \frac{n}{2}\Big)\nabla_4\nabla^m\nabla_4^l\Psi^G - \Delta\nabla^m\nabla_4^l\Psi^G = \sum_{\Psi_0^G}\psi\nabla^{m+1}\nabla_4^l\Psi_0^G + Err^{\Psi}_{ml}, \end{cases} \tag{1.29}$$

where the error terms Err^{Ψ}_{ml} are defined in detail in Proposition 5.1. Once again, the solutions satisfy the expansions (1.28) at $\{v = 0\}$. In Section 10, we prove estimates with initial data at $(-1, 1)$ controlled by Ξ_M, and the energy at $(-1, 0)$ controlling $\big\|\Sigma(\slashed{g}_0, \check{h})\big\|_M$. At top order, we bound:

$$v^2\big\|\nabla_4\nabla^M\nabla_4^{\frac{n-4}{2}}\alpha\big\|^2_{H^{1/2}(S_{-1,v})} + v\big\|\nabla^M\nabla_4^{\frac{n-4}{2}}\alpha\big\|^2_{H^{3/2}(S_{-1,v})} +$$
$$+\big\|\nabla^M\nabla_4^{\frac{n-4}{2}}\Psi^G\big\|^2_{H^{3/2}(S_{-1,v})} + v\big\|\nabla_4\nabla^M\nabla_4^{\frac{n-4}{2}}\Psi^G\big\|^2_{H^{1/2}(S_{-1,v})} + \dots$$

and the asymptotic quantities:

$$\big\|\mathcal{O}\big\|^2_{H^{M+1}(S_{-1,0})} + \big\|\mathfrak{h}\big\|^2_{H^{M+1}(S_{-1,0})}.$$

This step represents the essential part of the proof, since as in the previous section once we control the top order terms we can also obtain bounds for the remaining terms in $\big\|\Sigma(\slashed{g}_0, \check{h})\big\|_M$ and estimate the nonlinear error terms. The strategy is again to treat the system (1.29) as a linear system on the background obtained by restricting g to $\{u = -1\}$, with a general inhomogeneous term. We refer to the system (1.43) as the **second model system**, introduced in Section 5, and we explain below how we prove estimates for it in Section 8.

Geometric Littlewood-Paley projections. The analysis of the model systems needed for the top order estimates requires the use of Littlewood-Paley projections. These provide a robust way of constructing frequency dependent multipliers and defining fractional derivatives, including the $\log \nabla$ operator present in the definition of $\mathfrak{h}$. In dealing with the model systems we intend to use the same approach as in [Cic23]. The new difficulty is that the metric $\not{g}_v$ induced by the background on the spheres $S_v = \{u=-1\} \times \{v\} \times S^n$ has a nontrivial time dependence, compared to the case of de Sitter space. This motivates us to use the geometric Littlewood-Paley theory of [KR06], defined using the heat equation in Section 6. The LP projections used have standard properties, with additional difficulties arising from the fact that they are time dependent and do not satisfy exact orthogonality. We also use the methods of [KR06, KR05] to prove additional new results.

In particular, the LP projections allow us to define fractional derivatives. For example, the operator $\log \nabla$ is defined as:

$$(\log \nabla) F = \sum_{k \geq 0} P_k^2 F \cdot \log 2^k,$$

where in Section 6 we choose the projection operators P_k such that $\sum_k P_k^2 = I$.

Unlike the case of the linear wave equation on exact de Sitter space in [Cic23], we encounter additional difficulties since the LP projections only satisfy L^2-almost orthogonality. For any two families of LP projections P_k and P'_l we have according to [KR06]:

$$\left\| P_k P'_l F \right\|_{L^2} \lesssim 2^{-4|k-l|} \cdot \left\| F \right\|_{L^2}. \tag{1.30}$$

The LP projections satisfy the finite band property $\|P_k F\|_{L^2} \lesssim 2^{-k} \|\nabla \widetilde{P}_k F\|_{L^2}$, where $\widetilde{P}_k^2 = P_k$. For certain top order quantities, having a different projection operator on the RHS (i.e. the term $\nabla \widetilde{P}_k F$ instead of $\nabla P_k F$) is problematic for the following reason: naively using the fact that $\sum_l P_l^2 = I$ and (1.30) to bound $\nabla \widetilde{P}_k F$ creates dangerous terms with frequency higher than k. Therefore, instead of using the finite band property, for certain top order quantities we use the following novel refined Poincaré inequality for any $k \geq 0$ and $\delta > 0$:

$$\big\|P_k F\big\|_{L^2}^2 \lesssim \frac{1}{\delta} 2^{-2k} \big\|\nabla P_k F\big\|_{L^2}^2 + \delta \sum_{0 \leq l < k} 2^{-9k+7l} \big\|\nabla P_l F\big\|_{L^2}^2 + \delta^{-1} 2^{-4k} \big\|F\big\|_{L^2}^2. \tag{1.31}$$

The important feature of this inequality is that all the LP projection operators have the same symbol. Moreover, only the last term contains frequencies higher than k, but this is lower order due to the good 2^{-4k} weight.

Finally, we also prove bounds for the commutation error terms obtained due to the time dependence of the metric $\not{g}$. For example, for a certain projection operator $\underline{\widetilde{P}}_k$ defined in Section 6 in terms of P_k, we have:

$$\big\|[\nabla_4, P_k] F\big\|_{L^2} \lesssim \big\|\underline{\widetilde{P}}_k F\big\|_{L^2} + 2^{-k} \big\|F\big\|_{L^2}. \tag{1.32}$$

While the presence of projection operators with different symbols in this inequality cannot be avoided, we point out that both sides of the inequality can be summed for $k \geq 0$, which will prove to be essential later.

The first model system. In Section 5, we write the system (1.27) as a linear system on the background obtained by restricting the metric g to the null cone $\{u = -1\}$, with a general inhomogeneous term. We use the notation $\Phi_0 = \nabla_4^{\frac{n-4}{2}} \alpha$ and $\Phi_i = \nabla_4^{\frac{n-4}{2}} \Psi^G$, and we obtain with respect to the new time variable $\tau = \sqrt{v}$ the system:

$$\begin{cases} \nabla_\tau\big(\nabla_\tau \nabla^m \Phi_0\big) + \frac{1}{\tau} \nabla_\tau \nabla^m \Phi_0 - 4\Delta \nabla^m \Phi_0 = \psi \nabla^{m+1} \Phi + F_m^0 \\ \nabla_\tau\big(\nabla_\tau \nabla^m \Phi_i\big) + \frac{1}{\tau} \nabla_\tau \nabla^m \Phi_i - 4\Delta \nabla^m \Phi_i = \psi \nabla^{m+1} \Phi + F_m^i \\ \Phi_0 = 2\mathcal{O} \log \tau + h + O\big(\tau^2 |\log \tau|^2\big), \ \Phi_i = \Phi_i^0 + O\big(\tau^2 |\log \tau|^2\big). \end{cases} \tag{1.33}$$

where the covariant angular derivatives are with respect to the metric $\not{g}_\tau := \not{g}_{u=-1, v=\tau^2}$ induced on S_τ.

Remark 1.7. *We notice that in terms of the metric* $\widetilde{\not{g}}$ *from* (1.3) *we have* $\not{g}_\tau = \widetilde{\not{g}}(\log(2\tau))$. *For any asymptotically de Sitter space of the form* (1.3) *we consider the new time coordinate* $\tau = e^T/2$. *We point out that one can also recover the first model system by commuting the Einstein equations* (1.1) *$n/2$ times with the vectorfield* $\frac{1}{2\tau}\partial_\tau$. *A similar approach also holds for the second model system* (1.43).

The estimates needed at top order for the system (1.27) are proved at the level of the first model system in Section 7. We provide a detailed outline, and note the reader should use this section for assistance when reading the proof of Theorem 7.1. Our strategy is to adapt the approach in [Cic23] to the current setting by using the geometric LP theory instead. We first decompose the solution into its singular and regular parts, which we treat separately in all the estimates. We then explain how to prove lower order estimates and top order estimates, which complete the proof of Theorem 7.1.

Step 1. Decomposition of Φ_0. The first model system (1.33) has a favorable structure for proving estimates forward in time for the quantities $\Phi_1, \ldots, \Phi_I$ in terms of the asymptotic data at $\mathcal{I}^-$, since these quantities are regular at $\tau = 0$, according to their asymptotic expansions. While the equation for Φ_0 has the same favorable structure, we encounter difficulties since Φ_0 blows-up at $\mathcal{I}^-$ as $\log \tau$. We isolate this singular behavior by defining the singular component of Φ_0, which decouples from the rest of the system and can be treated separately. The remaining component of Φ_0 is regular at $\mathcal{I}^-$ and can be treated similarly to the other regular quantities.

For each $m \leq M$, we define the singular component $\big(\nabla^m \Phi_0\big)_Y$ to be the horizontal tensor solving:

$$\nabla_\tau\big(\nabla_\tau\big(\nabla^m\Phi_0\big)_Y\big) + \frac{1}{\tau}\nabla_\tau\big(\nabla^m\Phi_0\big)_Y - 4\Delta\big(\nabla^m\Phi_0\big)_Y = \psi\nabla\big(\nabla^m\Phi_0\big)_Y, \tag{1.34}$$

with asymptotic data given by:

$$\big(\nabla^m\Phi_0\big)_Y(\tau) = 2\nabla^m\mathcal{O}\log(\tau) + 2(\log\nabla)\nabla^m\mathcal{O} + O\big(\tau^2|\log(\tau)|^2\big), \tag{1.35}$$

$$\nabla_\tau\big(\nabla^m\Phi_0\big)_Y(\tau) = \frac{2\nabla^m\mathcal{O}}{\tau} + O\big(\tau|\log(\tau)|^2\big). \tag{1.36}$$

We prove the existence and uniqueness of the singular component in Section 7.1. We define the regular component:

$$\big(\nabla^m\Phi_0\big)_J = \nabla^m\Phi_0 - \big(\nabla^m\Phi_0\big)_Y.$$

Making the renormalization $\mathfrak{h}_m = \nabla^m h - 2(\log\nabla)\nabla^m\mathcal{O}$, we get that the regular component satisfies the equation:

$$\nabla_\tau\big(\nabla_\tau\big(\nabla^m\Phi_0\big)_J\big) + \frac{1}{\tau}\nabla_\tau\big(\nabla^m\Phi_0\big)_J - 4\Delta\big(\nabla^m\Phi_0\big)_J = \psi\nabla\big(\nabla^m\Phi_0\big)_J + \sum_{j=1}^{I}\psi\nabla^{m+1}\Phi_j + F_m^0, \tag{1.37}$$

with asymptotic data given by:

$$\big(\nabla^m\Phi_0\big)_J(\tau)=\mathfrak{h}_m+O\big(\tau^2|\log(\tau)|^2\big),\ \nabla_\tau\big(\nabla^m\Phi_0\big)_J(\tau)=O\big(\tau|\log(\tau)|^2\big)\text{ in }C^\infty(S^n).\tag{1.38}$$

Step 2. Lower order estimates. In order to prove the needed refined estimates for the solution at top order, it is essential to first prove in Section 7.2 energy estimates that are lower order in terms of angular derivatives. We treat separately the singular and regular quantities. For the regular quantities we note that the structure of (1.33) and the regularity at $\mathcal{I}^-$ allow us to use ∇_τ as a multiplier to prove in Proposition 7.3:

$$\sum_{m=0}^{M-1}\big\|\big(\nabla^m\Phi_0\big)_J\big\|^2_{H^1}+\sum_{i=1}^{I}\big\|\Phi_i\big\|^2_{H^M}\leq C_I\mathcal{D}_I+C_I\mathcal{F}_I(\tau).\tag{1.39}$$

For the singular component, the lower order version of (7.2) consists of proving in Proposition 7.2:

$$\sum_{m=0}^{M-1}\big\|\big(\nabla^m\Phi_0\big)_Y\big\|^2_{H^1}\leq C_I\mathcal{D}_I+C_I|\log\tau|^2\big\|\mathcal{O}\big\|^2_{H^{M+1}}.\tag{1.40}$$

To prove this, we further decompose for every $m<M$: $\big(\nabla^m\Phi_0\big)_Y=\big(\nabla^m\Phi_0\big)^1_Y+\big(\nabla^m\Phi_0\big)^2_Y$, where $\big(\nabla^m\Phi_0\big)^1_Y$ and $\big(\nabla^m\Phi_0\big)^2_Y$ are the solutions of (7.4) that satisfy the asymptotic expansions:

$$\begin{aligned}\big(\nabla^m\Phi_0\big)^1_Y(\tau)=&2\nabla^m\mathcal{O}\log(\tau)+O\big(\tau^2|\log(\tau)|^2\big),\\ \big(\nabla^m\Phi_0\big)^2_Y(\tau)=&2(\log\nabla)\nabla^m\mathcal{O}+O\big(\tau^2|\log(\tau)|^2\big).\end{aligned}$$

The desired lower order estimate for the singular component follows using the ∇_τ multiplier in the equations for $\big(\nabla^m\Phi_0\big)^1_Y/\log\tau$ and $\big(\nabla^m\Phi_0\big)^2_Y$, similarly to the approach in [Cic23].

Step 3. Top order estimates for the singular component. The singular component decouples from the rest of the system, so it can be treated independently of the regular quantities in Section 7.3. In Theorem 7.2, we prove the following estimate for $\tau\in(0,1]$, with an implicit constant depending only on M:

$$\tau^2\big\|\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|^2_{H^{1/2}(S_\tau)}+\tau^2\big\|\nabla\big(\nabla^M\Phi_0\big)_Y\big\|^2_{H^{1/2}(S_\tau)}\lesssim\big\|\mathcal{O}\big\|^2_{H^{M+1}},\tag{1.41}$$

$$\sum_{m=0}^{M}\big\|\big(\nabla^m\Phi_0\big)_Y\big\|^2_{H^1(S_\tau)}\lesssim\big(1+|\log\tau|^2\big)\big\|\mathcal{O}\big\|^2_{H^{M+1}}.\tag{1.42}$$

In order to obtain sharp estimates at top order, we must use the structure in the expansion of $\big(\nabla^M\Phi_0\big)_Y$, which can only be seen at the level of each LP projection. We have schematically for every $k \geq 0$:

$$P_k\big(\nabla^M\Phi_0\big)_Y(\tau) = 2P_k\nabla^M\mathcal{O}\log(2^k\tau) + l.o.t. + O\big(\tau^2|\log(\tau)|^2\big).$$

As in the case of the linear wave equation on de Sitter space studied in [Cic23], we prove that $P_k\big(\nabla^M\Phi_0\big)_Y$ satisfies similar asymptotics to the second Bessel function Y_0 in terms of the new time variable $t = 2^k\tau$. A quantitative version of this statement is proved in Section 7.3 using suitable energy estimates in the low frequency regime $\tau \leq 2^{-k-1}$, with data given by the asymptotic initial data, and the high frequency regime $\tau \in [2^{-k-1}, 1]$, with data at $\tau = 2^{-k-1}$ given by the solution in the low frequency regime. We remark that the asymptotic behavior and the frequency dependent time of transition between the two regimes are responsible for the improvement in regularity: at $\tau = 1$ we control $M + 3/2$ derivatives of the solution in terms of $M + 1$ derivatives of the asymptotic data.

The main new difficulties compared to [Cic23] arise from the fact that the geometric LP projections are time dependent and do not satisfy exact orthogonality, as explained above. Using bounds such as (1.32) implies the presence of different projection operators in the estimates. In the low frequency regime in Section 7.3.1, we can mostly avoid this issue using the structure of the error terms and the lower order estimates from Section 7.2. However, in the high frequency regime in Section 7.3.2, this issue creates commutation terms that cannot be bounded at the level of each LP projection. As a result, we must carefully use the structure of the error terms and sum the estimates obtained for each LP projection before being able to close our estimates in Section 7.3.3.

Step 4. Top order estimates for the regular components. In order to capture the specific behavior of the regular components at top order, we consider the equations satisfied by the projections $P_k\big(\nabla^M\Phi_0\big)_J$ and $P_k\nabla^M\Phi_i$, for all $1 \leq i \leq I$ and $k \geq 0$.

For each P_k projection we consider the low frequency regime $\tau \in (0, 2^{-k-1}]$ in Section 7.4.1. We propagate the L^2 bounds satisfied by the asymptotic data at $\mathcal{I}^-$ using ∇_τ as a multiplier to obtain schematically for $\tau \in (0, 2^{-k-1}]$:

$$\big\|P_k\big(\nabla^M\Phi_0\big)_J\big\|^2_{H^1} + \sum_{i=1}^{I}\big\|P_k\nabla^M\Phi_i\big\|^2_{H^1} + \dots$$

$$\lesssim \|P_k\mathfrak{h}_M\|_{H^1}^2+\sum_{i=1}^{I}\|P_k\nabla^M\Phi_i^0\|_{H^1}^2+\sum_{i=0}^{I}\int_0^\tau \frac{1}{2^k}\|P_kF_M^i\|_{L^2}^2 d\tau'+\dots,$$

where we refer the reader to Proposition 7.11 for the exact estimate proved, which includes additional good terms on the LHS and certain error terms on the RHS. Using the commutation estimates for LP projections, the error terms on the RHS that contain Φ have good 2^{-k} weights.

For each P_k projection we also consider the high frequency regime $\tau\in[2^{-k-1},1]$ in Section 7.4.2. We prove boundedness for the energy obtained by using the multiplier $2^k\tau\nabla_\tau$ to get schematically for $\tau\in[2^{-k-1},1]$:

$$2^k\tau\big\|P_k\big(\nabla^M\Phi_0\big)_J\big\|_{H^1}^2+\sum_{i=1}^{I}2^k\tau\big\|P_k\nabla^M\Phi_i\big\|_{H^1}^2+\dots\lesssim$$

$$\lesssim \big\|P_k\big(\nabla^M\Phi_0\big)_J\big\|_{H^1}^2\big|_{\tau=2^{-k-1}}+\sum_{i=1}^{I}\big\|P_k\nabla^M\Phi_i\big\|_{H^1}^2\big|_{\tau=2^{-k-1}}+$$

$$+\sum_{i=0}^{I}\int_{2^{-k-1}}^{\tau}2^k\tau'\big\|P_kF_M^i\big\|_{L^2}^2d\tau'+\int_{2^{-k-1}}^{\tau}(\tau')^3\big\|\underline{\widetilde{P}}_k\nabla\big(\nabla^M\Phi_0\big)_J\big\|_{L^2}^2d\tau'+\dots,$$

where we refer the reader to Proposition 7.12 for the exact estimate proved. It is essential that in this estimate we obtain a top order bulk term with favorable sign, which can be dropped from the estimate[1]. We also notice that to deal with the error terms we use refined LP commutation estimates such as (1.32). In particular, this creates error terms with different projection operators, such as the last term written above, which cannot be controlled directly at the level of the high frequency regime estimates.

To conclude the proof of the top order estimates for the regular components, we combine the high frequency regime and low frequency regime estimates in Section 7.4.3. We point out that this argument uses the top order estimates for the singular component, which are proved separately as we explain below. To bound the error terms containing different LP projection operators, we first need to sum the estimates obtained for all $k\geq 0$, and then use Grönwall. For the negative frequencies we use the lower order estimates from Section 7.2. As a result, we obtain:

$$\tau\big\|\big(\nabla^M\Phi_0\big)_J\big\|_{H^{3/2}}^2+\sum_{i=1}^{I}\tau\big\|\nabla^{M+1}\Phi_i\big\|_{H^{1/2}}^2+\dots\leq C_I\mathcal{D}_I+C_I\mathcal{F}_I(\tau).$$

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[1] On the other hand, we point out that in the context of the second model system, the bulk term obtained in the high frequency regime estimate for the singular component has an unfavorable sign, creating major difficulties.

The second model system. Similarly to the case of the first model system, in Section 5 we write the system (1.29) with respect to the new time variable $\tau = \sqrt{v}$ to obtain the second model system:

$$\begin{cases} \nabla_\tau\big(\nabla_\tau\nabla^m\Phi_0\big) + \frac{1}{\tau}\nabla_\tau\nabla^m\Phi_0 - 4\Delta\nabla^m\Phi_0 = \psi\nabla^{m+1}\Phi + F_m^0 \\ \nabla_\tau\big(\nabla_\tau\nabla^m\Phi_i\big) - \frac{1}{\tau}\nabla_\tau\nabla^m\Phi_i - 4\Delta\nabla^m\Phi_i = \sum_{j\neq 0}\psi\nabla^{m+1}\Phi_j + F_m^i \\ \Phi_0 = 2\mathcal{O}\log\tau + h + O\big(\tau^2|\log\tau|^2\big),\ \Phi_i = \Phi_i^0 + O\big(\tau^2|\log\tau|^2\big). \end{cases} \tag{1.43}$$

In Section 8, we obtain estimates for solutions of the second model system at $\tau \in (0,1)$ in terms of the solution at $\tau = 1$. Additionally, we also prove estimates for the asymptotic data at $\mathcal{I}^-$. We provide a detailed outline of the proof of Theorem 8.1, and refer the reader to this section for assistance when reading the proof in Section 8. Our strategy is once again to use the geometric LP theory and adapt the approach in [Cic23] to the current setting. We first explain how to prove estimates for the regular quantities. We then outline the lower order estimates and top order estimates for the singular quantities, and we also explain the estimates for the asymptotic data at $\mathcal{I}^-$, completing the proof of Theorem 8.1.

Step 1. Estimates for the regular quantities. The second model system (1.43) has a favorable structure for proving estimates backwards in time for the regular quantities $\Phi_1,\dots,\Phi_I$ in terms of the initial data at $\tau = 1$, which allows us to obtain top order estimates directly using the ∇_τ multiplier. Another key feature of the second model system is that the singular terms $\psi\nabla^{m+1}\Phi_0$ are absent from the RHS of the equations for the regular quantities. Thus, the equations for the regular quantities decouple from the singular quantities, and we can estimate them separately. We obtain in Section 8.1:

$$\sum_{i=1}^{I}\big\|\Phi_i\big\|^2_{H^{M+3/2}} + \sum_{i=1}^{I}\big\|\nabla_\tau\nabla^M\Phi_i\big\|^2_{H^{1/2}} + \ldots \leq C_{II}\mathcal{D}_{II} + C_{II}\mathcal{F}_{II}(\tau), \tag{1.44}$$

where we refer the reader to Proposition 8.1 for the exact estimate proved. Using the expansions satisfied by the regular quantities at $\tau = 0$, we obtain for the asymptotic data at $\mathcal{I}^-$:

$$\sum_{i=1}^{I}\big\|\Phi_i^0\big\|^2_{H^{M+1}} \leq C_{II}\mathcal{D}_{II} + C_{II}\mathcal{F}_{II}(0). \tag{1.45}$$

Step 2. Estimates for the singular quantities. The structure of the equation satisfied by the singular quantity Φ_0 creates significant new challenges compared to the case of the regular quantities. Additionally, we must prove estimates consistent with the expansion of Φ_0 at $\tau = 0$, and obtain suitable bounds for $\mathcal{O}$ and $\mathfrak{h}$.

We start with a lower order estimate proved in Section 8.2, which provides a preliminary bound for Φ_0:

$$\sum_{m=0}^{M} \tau^2 \big\|\nabla_\tau \nabla^m \Phi_0\big\|_{L^2}^2 + \tau^2 \big\|\Phi_0\big\|_{H^{M+1}}^2 + \tau \big\|\Phi_0\big\|_{H^M}^2 + \int_\tau^1 \tau' \big\|\Phi_0\big\|_{H^{M+1}}^2 d\tau' \leq C_{II}\mathcal{D}_{II} + C_{II}\mathcal{F}_{II}(\tau). \tag{1.46}$$

The main part of the argument consists of proving sharp estimates for the top order quantity $\xi = \nabla^M \Phi_0$. We prove estimates for each projection $P_k \nabla^M \Phi_0$, with $k \geq 0$. For this purpose, we fix a suitably large constant $x > 0$ and we consider separately the low frequency regime $k \geq x$ with $\tau \in [0, X2^{-k-1}]$, and the high frequency regime $k \geq x$ with $\tau \in [X2^{-k-1}, 1]$, where we denoted $X = 2^{x+1}$. We note already here that we can deal with the terms with $k < x$ using the preliminary estimate (1.46).

In Section 8.3, we consider the low frequency regime $k \geq x$, $\tau \in [0, X2^{-k-1}]$, and we prove a similar estimate to (1.46) for $P_k \nabla^M \Phi_0$:

$$\begin{aligned} &\tau^2 \big\|P_k \nabla_\tau \nabla^M \Phi_0\big\|_{L^2}^2 + \tau^2 \big\|\nabla P_k \nabla^M \Phi_0\big\|_{L^2}^2 \\ &\qquad \lesssim X^2 2^{-2k} \Big(\big\|P_k \nabla_\tau \nabla^M \Phi_0\big\|_{L^2}^2 + \big\|\nabla P_k \nabla^M \Phi_0\big\|_{L^2}^2\Big)\Big|_{\tau = X2^{-k-1}} + C_X 2^{-3k} \mathcal{F}_{II}(\tau) + \dots, \end{aligned}$$

where we refer the reader to Proposition 8.4 for the exact estimate proved. We note that the data terms at $\tau = X2^{-k-1}$ will be bounded later using the high frequency regime estimates.

In Section 8.4, we consider the high frequency regime $\tau \in [X2^{-k-1}, 1]$, and we prove a similar estimate to the one for the first model system. Using the multiplier $2^k \tau \nabla_\tau$, we get:

$$\begin{aligned} 2^k \tau \big\|P_k \nabla^M \Phi_0\big\|_{H^1}^2 + \dots \lesssim 2^k \big\|P_k \nabla^M \Phi_0\big\|_{H^1}^2\big|_{\tau=1} + \int_\tau^1 \frac{2^k}{(\tau')^2} \big\|P_k \nabla^M \Phi_0\big\|_{L^2}^2 d\tau' \\ + \int_\tau^1 2^k \tau' \big\|P_k F'_M\big\|_{L^2}^2 d\tau' + \dots, \end{aligned}$$

where we refer the reader to Proposition 8.5 for the exact estimate proved.

The main difference compared to the first model system is the presence of the second term on the RHS above, which is a *top order bulk term* with an *unfavorable sign*, creating significant difficulties. We improve the above high frequency regime estimate and deal with this bad term in Section 8.5, which represents the main technical part of the paper. We use the refined Poincaré inequality (1.31) to bound the error term:

$$\int_{\tau}^{1} \frac{2^k}{(\tau')^2} \big\|P_k \nabla^M \Phi_0\big\|_{L^2}^2 d\tau'. \tag{1.47}$$

As a result, we obtain a sum of error terms on the RHS in the low frequency regime and high frequency regime. We bound the high frequency regime error terms using the novel Grönwall-like inequality in Lemma 8.1. For the low frequency regime error terms we use the estimates of Section 8.3, which in turn create a sum of error terms that we bound using the discrete Grönwall inequality.

Moreover, in the above estimates we also have error terms with different projection operators, obtained by using LP commutation estimates such as (1.32). As before, we deal with these terms towards the end of our argument, when summing the estimates obtained for all $k \geq x$.

Finally, in Section 8.6 we combine the improved high frequency regime estimates of Section 8.5 with the low frequency regime estimates of Section 8.3 and the preliminary estimate (1.46), to obtain the main estimate for Φ_0 :

$$\tau^2 \big\|\Phi_0\big\|_{H^{M+3/2}}^2 + \tau^2 \big\|\nabla_\tau \nabla^M \Phi_0\big\|_{H^{1/2}}^2 + \ldots \leq C_{II} \mathcal{D}_{II} + C_{II} \mathcal{F}_{II}(\tau),$$

where the remaining terms on the LHS are precisely the terms containing Φ_0 in the definition of $\mathcal{E}_{II}$ in (8.1) and we refer the reader to (8.20) for the exact estimate proved. Using the estimate for the regular quantities (1.44) as well, we conclude the proof of the main estimate (8.2) in Theorem 8.1.

Step 3. Estimates for the asymptotic quantities. The final step in the proof of Theorem 8.1 is showing the estimates (8.3) and (8.4) for the asymptotic data at $\mathcal{I}^-$ in Section 8.7. In view of (1.45), to establish (8.3) we prove:

$$\|\mathcal{O}\|_{H^{M+1}}^2 \leq C_{II} \mathcal{D}_{II} + C_{II} \mathcal{F}_{II}(0). \tag{1.48}$$

This estimate for the obstruction tensor follows by taking the limit $\tau \to 0$ in the low frequency regime estimate above for each $k \geq x$, and using the expansion of Φ_0 at $\mathcal{I}^-$.

The estimate (8.4) for $\mathfrak{h}$ is more involved. We first notice that it suffices to show:

$$\sum_{k \geq x} 2^{2k} \big\|P_k \mathfrak{h}_M\big\|_{L^2}^2 \leq C_{II} \mathcal{D}_{II} + C_{II} \mathcal{F}_{II}(0), \tag{1.49}$$

where $\mathfrak{h}_M = \nabla^M h - 2(\log \nabla)\nabla^M \mathcal{O}$. According to the expansion of $\nabla^M \Phi_0$ at $\mathcal{I}^-$, we can prove the above bound using energy estimates for the equations satisfied by the quantities $\overline{\xi}_k = P_k \nabla^M \Phi_0 - \tau \log(2^k \tau) P_k \nabla_\tau \nabla^M \Phi_0$. To deal with the error terms obtained in this case, we use the previously established bounds for the low frequency regime and the high frequency regime.

1.4 Outline of the Thesis

We outline the structure of the thesis. Chapter 2 consists of Sections 2-4. In Section 2 we introduce the double null formalism adapted to the setting of straight self-similar spacetimes. In Section 3 we prove the existence and uniqueness of scattering states, establishing the first statement of Theorem 1.3. In Section 4 we prove asymptotic completeness, obtaining the second statement of Theorem 1.3. Chapter 3 consists of Sections 5-8. In Section 5 we introduce the model systems necessary for the top order estimates of the scattering map. In Section 6 we introduce the geometric Littlewood-Paley projections used in the analysis of the model systems. In Section 7 we prove the top order estimates for the first model system. In Section 8 we prove the top order estimates for the second model system. Chapter 4 consists of Sections 9-11. In Section 9 we prove sharp estimates from $\{v = 0\}$ to $\{v = -u\}$. In Section 10 to prove sharp estimates from to $\{v = -u\}$ to $\{v = 0\}$. Finally, in Section 11 we combine our results to conclude the proof of the third statement of Theorem 1.3.

Chapter 2

Existence, Uniqueness of Scattering States, and Asymptotic Completeness

2 Set Up

We introduce the double null formalism in an arbitrary dimension due to [RSR18] in Section 2.1. We explain the fundamental simplifications obtained in the setting of straight self-similar vacuum spacetimes in Section 2.2, which will play a crucial role in our analysis. We also introduce in Section 2.3 the commutation formulas and the error term notation for future reference. This section is based on [Cic24, Section 2].

2.1 Double Null Gauge

We introduce a double null gauge on the $(n+2)$-dimensional manifold $\big(\mathcal{M}, g\big)$ following the work of [RSR18, Section 3]. In this section, we consider such a foliation in the general setting, in order to define the relevant quantities and write down the system of Einstein vacuum equations (1.14) in double null gauge. In the next section, we will use the additional assumptions of self-similarity and straightness of the spacetime $\big(\mathcal{M}, g\big)$, which simplify our equations significantly. We assume for the purpose of this section that g is smooth, and we later define the notion of regular vacuum solutions in Definition 2.3. Our introduction of the double null gauge will be brief, and we encourage the reader to consult [RSR18, Section 3] for complete statements and proofs.

We assume the background differentiable manifold to be $\big((-\infty,0]\times[0,\infty)\backslash\{(0,0)\}\big)\times S^n$, where the coordinates (u,v) parameterize $(-\infty,0]\times[0,\infty)\backslash\{(0,0)\}$. We consider the metric g in double null gauge:

$$g=-2\Omega^2(du\otimes dv+dv\otimes du)+\slashed{g}_{AB}(d\theta^A-b^A du)\otimes(d\theta^B-b^B du), \tag{2.1}$$

where $\{\theta^A\}$ represent local coordinates on S^n. We denote $S_{u,v}=\{u\}\times\{v\}\times S^n$. We define the normalized frame:

$$e_3=\Omega^{-1}(\partial_u+b^A\partial_A),\ e_4=\Omega^{-1}\partial_v,\ g(e_3,e_4)=-2. \tag{2.2}$$

We denote by D the Levi-Civita connection of $\big(\mathcal{M},g\big)$. Following [RSR18, Section 3] and the references therein, we introduce the notion of horizontal tensors on $S_{u,v}$, with entries in the tangent space of $S_{u,v}$, and we denote by $\nabla_A,\ \nabla_3,\ \nabla_4$ the projections of $D_A,\ D_3,\ D_4$ to the tangent space of $S_{u,v}$, for any vector e_A tangent to $S_{u,v}$.

We define the Ricci coefficients denoted schematically by $\psi\in\{\chi,\underline{\chi},\eta,\underline{\eta},\omega,\underline{\omega},\zeta\}$ as:

$$\chi_{AB}=g(D_Ae_4,e_B),\ \underline{\chi}_{AB}=g(D_Ae_3,e_B),\ \eta_A=-\frac{1}{2}g(D_3e_A,e_4),\ \underline{\eta}_A=-\frac{1}{2}g(D_4e_A,e_3)$$
$$\omega=-\frac{1}{4}g(D_4e_3,e_4),\ \underline{\omega}=-\frac{1}{4}g(D_3e_4,e_3),\ \zeta_A=\frac{1}{2}g(D_Ae_4,e_3).$$

We decompose the null second fundamental forms χ and $\underline{\chi}$ into their trace-free and trace parts:

$$\chi_{AB}=\hat{\chi}_{AB}+\frac{1}{n}\mathrm{tr}\chi\slashed{g}_{AB},\ \underline{\chi}_{AB}=\underline{\hat{\chi}}_{AB}+\frac{1}{n}\mathrm{tr}\underline{\chi}\slashed{g}_{AB}. \tag{2.3}$$

According to [RSR18, Lemma 3.2], we have the Ricci formulas:

$$D_4e_4=-2\omega e_4,\ D_4e_3=2\omega e_3+2\underline{\eta}^Ae_A,\ D_4e_A=\underline{\eta}_Ae_4+\nabla_4e_A \tag{2.4}$$

$$D_3e_3=-2\underline{\omega}e_3,\ D_3e_4=2\underline{\omega}e_4+2\eta^Ae_A,\ D_3e_A=\eta_Ae_3+\nabla_3e_A \tag{2.5}$$

$$D_Ae_4=-\zeta_Ae_4+\chi_A^Be_B,\ D_Ae_3=\zeta_Ae_3+\underline{\chi}_A^Be_B,\ D_Ae_B=\frac{1}{2}\underline{\chi}_{AB}e_4+\frac{1}{2}\chi_{AB}e_3+\nabla_Ae_B. \tag{2.6}$$

According to [RSR18, Proposition 3.1], we also have the metric equations:

$$\mathcal{L}_4\slashed{g}_{AB}=2\chi_{AB},\ \mathcal{L}_3\slashed{g}_{AB}=2\underline{\chi}_{AB},\ \omega=-\frac{1}{2}\nabla_4\log\Omega,\ \underline{\omega}=-\frac{1}{2}\nabla_3\log\Omega \tag{2.7}$$

$$\zeta_A=-\frac{1}{4}\Omega^{-1}\slashed{g}_{AB}e_4(b^B),\ \eta_A=\zeta_A+\nabla_A\log\Omega,\ \underline{\eta}_A=-\zeta_A+\nabla_A\log\Omega. \tag{2.8}$$

We define the curvature components[1] denoted schematically by $\Psi \in \{\alpha, \underline{\alpha}, \beta, \underline{\beta}, \nu, \underline{\nu}, \sigma, \rho, \tau\}$:

$$\alpha_{AB} = R_{A4B4},\ \underline{\alpha}_{AB} = R_{A3B3},\ \beta_A = \frac{1}{2}R_{A434},\ \underline{\beta}_A = \frac{1}{2}R_{A334},\ \nu_{ABC} = R_{ABC4},\ \underline{\nu}_{ABC} = R_{ABC3}$$

$$\sigma_{AB} = \frac{1}{2}R_{3A4B} - \frac{1}{2}R_{3B4A},\ \tau_{AB} = \frac{1}{2}R_{3A4B} + \frac{1}{2}R_{3B4A},\ \rho = \frac{1}{4}R_{4343}.$$

According to [RSR18, Lemma 3.1], we have the formulas:

$$\mathrm{tr}\alpha = Ric_{44},\ \mathrm{tr}\underline{\alpha} = Ric_{33},\ \mathrm{tr}\tau = Ric_{34} - 2\rho$$

$$\tau_{AB} = \not{g}^{CD}R_{CADB} - Ric_{AB},\ \beta_A = {\nu_{AB}}^{B} + Ric_{A4},\ \underline{\beta}_A = -{\underline{\nu}_{AB}}^{B} - Ric_{A3}.$$

The Einstein vacuum equations for $(\mathcal{M}, g)$ are equivalent to a set of null structure equations and constraint equations involving the Ricci coefficients and the curvature components. We state the null structure equations according to [RSR18], and we refer the reader to [RSR18, Section 3.5] for the derivation of the equations.

Proposition 2.1. *[RSR18, Proposition 3.2] We have the null structure equations for the vacuum spacetime* $(\mathcal{M}, g)$*:*

$$\nabla_4 tr\chi + \frac{1}{n}\big(tr\chi\big)^2 = -|\hat{\chi}|^2 - 2\omega tr\chi$$

$$\nabla_4 \hat{\chi}_{AB} + \frac{2}{n}tr\chi\hat{\chi}_{AB} = -\alpha_{AB} - 2\omega\hat{\chi}_{AB} + \hat{\chi}\cdot\hat{\chi}$$

$$\nabla_3 tr\underline{\chi} + \frac{1}{n}\big(tr\underline{\chi}\big)^2 = -|\underline{\hat{\chi}}|^2 - 2\underline{\omega}tr\underline{\chi}$$

$$\nabla_3 \underline{\hat{\chi}}_{AB} + \frac{2}{n}tr\underline{\chi}\underline{\hat{\chi}}_{AB} = -\underline{\alpha}_{AB} - 2\underline{\omega}\underline{\hat{\chi}}_{AB} + \underline{\hat{\chi}}\cdot\underline{\hat{\chi}}$$

$$\nabla_3 \hat{\chi}_{AB} + \frac{1}{n}tr\underline{\chi}\hat{\chi}_{AB} = -\hat{\tau}_{AB} + 2\underline{\omega}\hat{\chi}_{AB} + \big(\nabla\hat{\otimes}\eta\big)_{AB} - \frac{1}{n}tr\chi\underline{\hat{\chi}}_{AB} + \eta\cdot\eta + \hat{\chi}\cdot\underline{\hat{\chi}}$$

$$\nabla_3 tr\chi + \frac{1}{n}tr\underline{\chi}tr\chi = 2\rho + 2\underline{\omega}tr\chi + 2div(\eta) + 2|\eta|^2 + \hat{\chi}\cdot\underline{\hat{\chi}}$$

$$\nabla_4 \underline{\hat{\chi}}_{AB} + \frac{1}{n}tr\chi\underline{\hat{\chi}}_{AB} = -\hat{\tau}_{AB} + 2\omega\underline{\hat{\chi}}_{AB} + \big(\nabla\hat{\otimes}\underline{\eta}\big)_{AB} - \frac{1}{n}tr\underline{\chi}\hat{\chi}_{AB} + \underline{\eta}\cdot\underline{\eta} + \hat{\chi}\cdot\underline{\hat{\chi}}$$

$$\nabla_4 tr\underline{\chi} + \frac{1}{n}tr\underline{\chi}tr\chi = 2\rho + 2\omega tr\underline{\chi} + 2div(\underline{\eta}) + 2|\underline{\eta}|^2 + \hat{\chi}\cdot\underline{\hat{\chi}}$$

$$\nabla_4\eta = -\beta + \chi\cdot(\eta - \underline{\eta}) \tag{2.9}$$

$$\nabla_3\underline{\eta} = \underline{\beta} + \underline{\chi}\cdot(\eta - \underline{\eta}) \tag{2.10}$$

$$\nabla_4\underline{\omega} = \frac{1}{2}\rho + \frac{1}{4}|\underline{\eta}|^2 - \frac{1}{4}|\eta|^2 + 2\omega\underline{\omega} + 3|\zeta|^2 - |\nabla\log\Omega|^2 \tag{2.11}$$

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[1] We point out that in Sections 5.2,7, and 8 we will use τ to denote a time coordinate, but it will be clear from the context throughout the thesis when τ is used as a curvature component.

$$\nabla_3 \omega = \frac{1}{2}\rho - \frac{1}{4}|\underline{\eta}|^2 + \frac{1}{4}|\eta|^2 + 2\omega\underline{\omega} + 3|\zeta|^2 - |\nabla \log \Omega|^2. \tag{2.12}$$

where $\psi \cdot \psi$ is a schematic notation for certain contractions of Ricci coefficient terms.

Next, we state the constraint equations according to [RSR18], and we refer the reader to [RSR18, Section 3.6] for the derivation of the equations.

Proposition 2.2. *[RSR18, Propositions 3.3,3.5] We have the constraint equations for the vacuum spacetime $\big(\mathcal{M}, g\big)$:*

$$\begin{aligned}
Ri\not{e}m_{ABCD} &= R_{ABCD} + \frac{1}{2}\Big(\underline{\chi}_{BC}\chi_{AD} + \underline{\chi}_{AD}\chi_{BC} - \underline{\chi}_{AC}\chi_{BD} - \underline{\chi}_{BD}\chi_{AC}\Big) \\
R\not{i}c_{AB} &= \tau_{AB} - \frac{1}{2}tr\chi\underline{\chi}_{AB} - \frac{1}{2}tr\underline{\chi}\chi_{AB} + \chi^{C}_{(A}\underline{\chi}_{B)}C \\
\not{R} &= -2\rho + \frac{1-n}{n}tr\chi tr\underline{\chi} + \hat{\chi}\cdot\hat{\underline{\chi}} \\
\nabla_A\chi_{BC} - \nabla_B\chi_{AC} &= \nu_{ABC} + \chi_{AC}\zeta_B - \chi_{BC}\zeta_A \\
\nabla_A\underline{\chi}_{BC} - \nabla_B\underline{\chi}_{AC} &= \underline{\nu}_{ABC} - \underline{\chi}_{AC}\zeta_B + \underline{\chi}_{BC}\zeta_A \\
\nabla^A\chi_{AB} - \nabla_B tr\chi &= -\beta_B + tr\chi\zeta_B - \zeta^A\chi_{AB} \\
\nabla^A\underline{\chi}_{AB} - \nabla_B tr\underline{\chi} &= \underline{\beta}_B - tr\underline{\chi}\zeta_B + \zeta^A\underline{\chi}_{AB} \\
\nabla_A\eta_B - \nabla_B\eta_A = -\nabla_A\underline{\eta}_B + \underline{\nabla}_B\eta_A &= \sigma_{AB} + \frac{1}{2}\Big(\hat{\underline{\chi}}^{C}_{A}\hat{\chi}_{CB} - \hat{\underline{\chi}}^{C}_{B}\hat{\chi}_{CA}\Big) \qquad (2.13) \\
\nabla^A R_{ABCD} &= 2\nabla_{[C}\tau_{D]B} + \chi\cdot\underline{\nu} + \underline{\chi}\cdot\nu + \zeta\cdot\chi\cdot\underline{\chi}.
\end{aligned}$$

2.2 Self-Similar Straight Vacuum Spacetimes

The $(n+2)$-dimensional vacuum spacetimes $\big(\mathcal{M}, g\big)$ are straight and self-similar, according to Definition 1.1. In this section we derive the consequences of these properties, which simplify the previous null structure equations and constraint equations. These simplifications are essential for our analysis and play a fundamental role for the rest of the work. We also write down the system of Bianchi equations in the case of self-similar straight vacuum spacetimes. Finally, we introduce the notion of regular solutions, following [RSR18, Section 3].

We assume that $\big(\mathcal{M}, g\big)$ is self-similar, so for $S = u\partial_u + v\partial_v$ we have:

$$\mathcal{L}_S g = 2g. \tag{2.14}$$

We also assume that $\big(\mathcal{M}, g\big)$ is a straight spacetime, so according to Definition 1.1 it satisfies:

$$\Omega^2 = 1,\;\; b = 0. \tag{2.15}$$

Thus, the metric (2.1) in double null gauge takes the special form:

$$g = -2(du \otimes dv + dv \otimes du) + \not{g}_{AB} d\theta^A \otimes d\theta^B. \tag{2.16}$$

Moreover, the frame $\{e_A,\; e_3,\; e_4\}$ is integrable, and (2.2) becomes:

$$e_3 = \partial_u,\;\; e_4 = \partial_v,\;\; e_A = \partial_{\theta^A}. \tag{2.17}$$

As a consequence, we obtain that for self-similar straight vacuum spacetimes some of the Ricci coefficients and curvature components vanish identically:

Proposition 2.3. *Let* $\big(\mathcal{M}, g\big)$ *be a self-similar straight vacuum spacetime in double null gauge* (2.16)*. The nontrivial Ricci coefficients are* $\psi \in \{tr\chi, \hat{\chi}, tr\underline{\chi}, \hat{\underline{\chi}}\}$*, and the nontrivial curvature components are* $\Psi \in \{\alpha, \nu, \tau, R, \underline{\nu}, \underline{\alpha}\}$*. More specifically, we have the following equations:*

$$\begin{aligned} &u\underline{\chi}_{AB} + v\chi_{AB} = \not{g}_{AB} \\ &\eta = \underline{\eta} = \zeta = 0,\;\; \omega = \underline{\omega} = 0 \\ &\sigma = 0,\;\; \rho = 0,\;\; \beta = \underline{\beta} = 0. \end{aligned} \tag{2.18}$$

Proof. Equation (2.18) is proved in [RSR18, Appendix B] as a consequence of (2.14) and (2.15). The metric equations (2.7)-(2.8), together with $\Omega = 1,\; b = 0$, imply the vanishing of the Ricci coefficients $\eta, \underline{\eta}, \zeta, \omega, \underline{\omega}$. The null structure equations for (2.9)-(2.12) imply $\beta = \underline{\beta} = 0$, and $\rho = 0$. Finally, the constraint equation (2.13) implies that:

$$\sigma_{AB} = \frac{1}{2}\Big(-\hat{\underline{\chi}}_A^C \hat{\chi}_{CB} + \hat{\underline{\chi}}_B^C \hat{\chi}_{CA}\Big).$$

However, we also have from (2.18) that $-u\hat{\underline{\chi}}_{AB} = v\hat{\chi}_{AB}$, which then gives $\sigma = 0$. □

We simplify the null structure equations in Proposition 2.1 and the constraint equations in Proposition 2.2 using the vanishing properties proved in Proposition 2.3. In particular, the null structure equations (2.9)-(2.12) are trivial. Moreover, we notice that all the terms containing angular derivatives in the null structure equations vanish. As a result, when treating these equations as a system of transport equations we avoid the complications regarding the loss of angular derivatives which one usually faces for a general metric in double null gauge.

We record here the simplified null structure equations:

Proposition 2.4. *We have the null structure equations for the self-similar straight vacuum spacetime* $\big(\mathcal{M}, g\big)$*:*

$$\nabla_4 tr\chi + \frac{1}{n}\big(tr\chi\big)^2 = -|\hat{\chi}|^2,\ \nabla_3 tr\chi + \frac{1}{n} tr\underline{\chi} tr\chi = \hat{\chi}\cdot\underline{\hat{\chi}}$$

$$\nabla_4\hat{\chi}_{AB} + \frac{2}{n} tr\chi\hat{\chi}_{AB} = -\alpha_{AB} + \hat{\chi}\cdot\hat{\chi},\ \nabla_3\hat{\chi}_{AB} + \frac{1}{n} tr\underline{\chi}\hat{\chi}_{AB} = -\hat{\tau}_{AB} - \frac{1}{n} tr\chi\underline{\hat{\chi}}_{AB} + \hat{\chi}\cdot\underline{\hat{\chi}}$$

$$\nabla_3 tr\underline{\chi} + \frac{1}{n}\big(tr\underline{\chi}\big)^2 = -|\underline{\hat{\chi}}|^2,\ \nabla_4 tr\underline{\chi} + \frac{1}{n} tr\underline{\chi} tr\chi = \hat{\chi}\cdot\underline{\hat{\chi}}$$

$$\nabla_3\underline{\hat{\chi}}_{AB} + \frac{2}{n} tr\underline{\chi}\underline{\hat{\chi}}_{AB} = -\underline{\alpha}_{AB} + \underline{\hat{\chi}}\cdot\underline{\hat{\chi}},\ \nabla_4\underline{\hat{\chi}}_{AB} + \frac{1}{n} tr\chi\underline{\hat{\chi}}_{AB} = -\hat{\tau}_{AB} - \frac{1}{n} tr\underline{\chi}\hat{\chi}_{AB} + \hat{\chi}\cdot\underline{\hat{\chi}},$$

where $\psi\cdot\psi$ *is a schematic notation for certain contractions of Ricci coefficient terms.*

We also record here the simplified constraint equations:

Proposition 2.5. *We have the constraint equations for the self-similar straight vacuum spacetime* $\big(\mathcal{M}, g\big)$*:*

$$Ri\not{e}m_{ABCD} = R_{ABCD} + \frac{1}{2}\Big(\underline{\chi}_{BC}\chi_{AD} + \underline{\chi}_{AD}\chi_{BC} - \underline{\chi}_{AC}\chi_{BD} - \underline{\chi}_{BD}\chi_{AC}\Big) \tag{2.19}$$

$$R\not{i}c_{AB} = \tau_{AB} - \frac{1}{2} tr\chi\underline{\chi}_{AB} - \frac{1}{2} tr\underline{\chi}\chi_{AB} + \chi^{C}_{(A}\underline{\chi}_{B)}C,\ \not{R} = \frac{1-n}{n} tr\chi tr\underline{\chi} + \hat{\chi}\cdot\underline{\hat{\chi}}$$

$$\nabla_A\chi_{BC} - \nabla_B\chi_{AC} = \nu_{ABC},\ \nabla_A\underline{\chi}_{BC} - \nabla_B\underline{\chi}_{AC} = \underline{\nu}_{ABC}$$

$$\nabla^A\chi_{AB} = \nabla_B tr\chi,\ \nabla^A\underline{\chi}_{AB} = \nabla_B tr\underline{\chi}$$

$$\nabla^A R_{ABCD} = 2\nabla_{[C}\tau_{D]B} + \chi\cdot\underline{\nu} + \underline{\chi}\cdot\nu + \zeta\cdot\chi\cdot\underline{\chi}. \tag{2.20}$$

We note some further consequences of the self-similarity and straightness properties:

Lemma 2.1. *The frame* $\{e_A, e_3, e_4\}$ *defined in* (2.17) *satisfies:*

$$\nabla_4 e_A = \frac{1}{v} e_A - \frac{u}{v}\underline{\chi}_A^B e_B,\ \nabla_3 e_A = \underline{\chi}_A^B e_B. \tag{2.21}$$

Proof. Using Proposition 2.3 in the Ricci formulas (2.4)-(2.6), we have:

$$\nabla_4 e_A = D_4 e_A = D_A e_4 = \chi_A^B e_B = \frac{1}{v} e_A - \frac{u}{v}\underline{\chi}_A^B e_B,\ \nabla_3 e_A = D_A e_3 = \underline{\chi}_A^B e_B.$$

□

Lemma 2.2. *For any curvature component* Ψ *we have that* $\nabla_S\Psi = -2\Psi$*.*

Proof. For $\Psi \in \{\alpha, \tau, \underline{\alpha}\}$, we have that $\mathcal{L}_S \Psi_{AB} = 0$ implies that $u\partial_u(\Psi_{AB}) + v\partial_v(\Psi_{AB}) = 0$ in the canonical coordinate frame. Equation (2.21) then implies that $\nabla_S \Psi_{AB} = -2\Psi_{AB}$. The proof is similar in the case when $\Psi \in \{\nu, \underline{\nu}\}$, for which we have $u\partial_u(\Psi_{ABC}) + v\partial_v(\Psi_{ABC}) = \Psi_{ABC}$ and for $\Psi = R$, for which we have $u\partial_u(R_{ABCD}) + v\partial_v(R_{ABCD}) = 2R_{ABCD}$. $\square$

In our analysis, it is convenient to simplify notation by representing error terms schematically. An essential tool for keeping track of error terms is the notion of signature of [RSR18, Section 3.3]:

Definition 2.1. *For any $\phi \in \{\psi, \Psi\}$ we define the signature:*

$$s(\phi) = N_3(\phi) + \frac{1}{2}N_A(\phi) - 1,$$

where N_3 represents the number of e_3 vectors used in the definition of ϕ, and N_A represents the number of e_A vectors used in the definition of ϕ.

We state the results for signature in [RSR18, Section 3.3] in the case of self-similar straight spacetimes:

Lemma 2.3. *[RSR18, Section 3.3] The signatures of the nontrivial Ricci coefficients and curvature components for a self-similar straight vacuum spacetime are:*

$$s(\chi) = 0,\ s(\underline{\chi}) = 1,\ s(\alpha) = 0,\ s(\nu) = \frac{1}{2},\ s(\tau) = s(R) = 1\ s(\underline{\nu}) = \frac{3}{2},\ s(\underline{\alpha}) = 2.$$

Moreover, we have that for any horizontal tensors ϕ, ϕ_1, ϕ_2 :

$$s(\nabla_3\phi) = s(\phi) + 1,\ s(\nabla_A\phi) = s(\phi) + \frac{1}{2},\ s(\nabla_4\phi) = s(\phi),\ s(\phi_1\phi_2) = s(\phi_1) + s(\phi_2).$$

Thus, the signature is preserved by covariant differentiation.

We use this notion of signature to introduce the schematic notation of [RSR18, Section 3.8] for the error terms that we expect on the right hand side of the Bianchi equations:

Definition 2.2. *[RSR18, Definition 3.2] For any $s \in \{0, \frac{1}{2}, \dots, \frac{5}{2}\}$ we define the schematic error terms:*

$$\mathcal{E}_s^{(3)} = \sum_{s_1+s_2=s,\ s_1\neq 1} \psi_{s_1}\Psi_{s_2},\ \mathcal{E}_s^{(4)} = \sum_{s_1+s_2=s} \psi_{s_1}\Psi_{s_2},$$

where ψ_{s_1} denotes a Ricci coefficient with signature s_1 and Ψ_{s_2} denotes a curvature component with signature s_2.

Remark 2.1. *We point out that in [RSR18, Definition 3.2], the schematic error terms also contain cubic terms of the form* $\zeta \sum_{s_1+s_2=s-1/2} \psi_{s_1}\psi_{s_2}$*. However, since* $\zeta = 0$ *according to Proposition 2.3, these terms are absent in the case of self-similar straight spacetimes, which further simplifies the structure of the error terms that we encounter.*

We can now state the Bianchi equations for self-similar straight spacetimes. We point out that the reader should consult [RSR18, Proposition 3.6] for the general system of Bianchi equations for vacuum metrics in double null gauge (2.1), which for the sake of exposition we do not repeat here. In our setting, due to the vanishing of certain Ricci coefficients and curvature components, the Bianchi equations in [RSR18, Proposition 3.6] imply two sets of equations:

- Simplified Bianchi equations satisfied by the nontrivial curvature components α, ν, $\tau, R, \underline{\nu}$, and $\underline{\alpha}$.
- Constrain equations, obtained from the Bianchi equations for the vanishing curvature components $\beta, \underline{\beta}, \sigma$.

As a consequence of [RSR18, Proposition 3.6] and Proposition 2.3, we obtain the following simplified Bianchi equations:

Proposition 2.6. *We have the following Bianchi equations for the straight self-similar vacuum spacetime* $(\mathcal{M}, g)$*:*

$$\nabla_3\alpha_{AB} + \frac{1}{2}tr\underline{\chi}\alpha_{AB} = -\nabla^C\nu_{C(AB)} + \mathcal{E}_1^{(3)} \tag{2.22}$$

$$\nabla_4\nu_{ABC} = -2\nabla_{[A}\alpha_{B]C} + \mathcal{E}_{1/2}^{(4)} \tag{2.23}$$

$$\nabla_3\nu_{ABC} + \frac{2}{n}tr\underline{\chi}\nu_{ABC} = -2\nabla_{[A}\tau_{B]C} + 2\hat{\underline{\chi}}_{[A}^{D}\nu_{|D|B]C} + \mathcal{E}_{3/2}^{(3)} \tag{2.24}$$

$$\nabla_4 R_{ABCD} = -2\nabla_{[A}\nu_{|CD|B]} + \mathcal{E}_1^{(4)} \tag{2.25}$$

$$\nabla_3 R_{ABCD} + \frac{2}{n}tr\underline{\chi}R_{ABCD} = -2\nabla_{[A}\underline{\nu}_{|CD|B]} + \underline{\chi}_{A[D}\tau_{C]B} + \underline{\chi}_{B[C}\tau_{D]A} + 2\hat{\underline{\chi}}_{[A}^{E}R_{B]ECD} + \mathcal{E}_2^{(3)} \tag{2.26}$$

$$\nabla_4\underline{\nu}_{ABC} = -2\nabla_{[A}\tau_{B]C} + \mathcal{E}_{3/2}^{(4)} \tag{2.27}$$

$$\nabla_3\underline{\nu}_{ABC} + \frac{3}{n}tr\underline{\chi}\underline{\nu}_{ABC} = -2\nabla_{[A}\underline{\alpha}_{B]C} + 2\hat{\underline{\chi}}_{[A}^{D}\underline{\nu}_{B]DC} + 2\hat{\underline{\chi}}_{[A}^{D}\underline{\nu}_{|CD|B]} + \mathcal{E}_{5/2}^{(3)} \tag{2.28}$$

$$\nabla_4\underline{\alpha}_{AB} = -\nabla^C\underline{\nu}_{C(AB)} + \mathcal{E}_2^{(4)}, \tag{2.29}$$

where one can also derive equations for τ *using* $\tau_{AB} = \not{g}^{CD} R_{CADB}$.

Moreover, as a consequence of the Bianchi equations in [RSR18, Proposition 3.6] for the vanishing curvature components $\{\beta, \underline{\beta}, \sigma\}$ and Proposition 2.3, we obtain the following additional constraint equations:

Proposition 2.7. *We have the following constraint equations for the straight self-similar vacuum spacetime* $\big(\mathcal{M}, g\big)$*:*

$$\nabla^B \alpha_{AB} = \mathcal{E}^{(4)}_{1/2},\ \nabla^C \nu_{ABC} = \mathcal{E}^{(4)}_{1},\ \nabla^C \underline{\nu}_{ABC} = \mathcal{E}^{(4)}_{2},\ \nabla^B \underline{\alpha}_{AB} = \mathcal{E}^{(4)}_{5/2}. \tag{2.30}$$

We conclude this section by defining the notion of a regular solution to the Einstein vacuum equations (1.14), according to [RSR18, Section 3.10]. We refer to the collection of horizontal tensors $\not{g},\ \chi,\ \underline{\chi},\ \alpha,\ \nu,\ \tau,\ R,\ \underline{\nu},\ \underline{\alpha}$ introduced above as the set of double null unknowns.

Definition 2.3. *[RSR18, Definition 3.5] The straight self-similar metric* g *defined on a subset of the background differentiable manifold* $\big((-\infty, 0] \times [0, \infty) \backslash \{(0,0)\}\big) \times S^n$, *is a regular solution of* (1.14) *if:*

- *For* $n > 4$ *even, the metric:*

  $$g = -2(du \otimes dv + dv \otimes du) + \not{g}_{AB} d\theta^A \otimes d\theta^B$$

  is a classical C^2 *solution of the Einstein vacuum equations* (1.14)*. Equivalently, the double null unknowns are classical solutions to the metric equations* (2.7)*-*(2.8)*, the system of null structure equations in Proposition 2.4, the Bianchi equations in Proposition 2.6, and the system of constraint equations in Propositions 2.5 and 2.7.*

- *For* $n = 4$ *the metric* g *defined as above is a classical* C^2 *solution of the Einstein vacuum equations* (1.14) *for* $v > 0$ *and* $u < 0$*. Moreover, the corresponding double null unknowns are classical solutions to the metric equations* (2.7)*-*(2.8)*, the system of constraint equations in Proposition 2.5 and weak solutions to the system of equations in Propositions 2.4, 2.6, and 2.7.*

The solutions that we construct in this work will be smooth in the region $\{v > 0,\ u < 0\}$ and extend as regular solutions to $\big((-\infty, 0] \times [0, \infty) \backslash \{(0,0)\}\big) \times S^n$. Moreover, the solutions will also satisfy the Fefferman–Graham expansions at $v = 0$ and $u = 0$ up to order $\frac{n}{2}$. In the case of $n = 4$, all the double null unknowns extend continuously to $v = 0$ and $u = 0$, with the exception of α which has a $\log v$ singularity at $v = 0$ and $\underline{\alpha}$ which has a $\log u$ singularity at $u = 0$. These mild singularities allow us to conclude that for $n = 4$ a regular solution g solves the Einstein vacuum equations (1.14) weakly in L^2.

2.3 Commutation Formulas and Error Terms Notation

In this section, we introduce the commutation formulas and the schematic error term notation that will play an essential role in our analysis and facilitate keeping track of nonlinear terms.

Lemma 2.4. *For any horizontal tensor ϕ, we have the commutation formulas:*

$$\big[\nabla_4, \nabla^i\big]\phi + \frac{i}{n} tr\chi \nabla^i \phi = \sum_{j=1}^{i} \nabla^j \chi \cdot \nabla^{i-j}\phi + \hat{\chi} \cdot \nabla^i \phi,$$

$$\big[\nabla_3, \nabla^i\big]\phi + \frac{i}{n} tr\underline{\chi} \nabla^i \phi = \sum_{j=1}^{i} \nabla^j \underline{\chi} \cdot \nabla^{i-j}\phi + \underline{\hat{\chi}} \cdot \nabla^i \phi,$$

where ∇^j is a schematic notation for all the possible combinations of j angular derivatives.

Proof. We prove the first statement, since the second one statement is similar. A standard computation, see for example [Luk12, Section 4.4], together with the additional vanishing of certain Ricci coefficients, implies that:

$$\big[\nabla_4, \nabla_A\big]\phi = \big[D_4, D_A\big]\phi - \chi_A^B \nabla_B \phi = -\chi_A^B \nabla_B \phi + \nu \cdot \phi = -\chi_A^B \nabla_B \phi + \nabla \chi \cdot \phi.$$

We can rewrite this as $\big[\nabla_4, \nabla_A\big]\phi + \frac{1}{n}\mathrm{tr}\chi \nabla_A \phi = \nabla \chi \cdot \phi + \hat{\chi} \cdot \nabla \phi$, and conclude by induction. □

Lemma 2.5. *For any horizontal tensor ϕ, we have the commutation formulas:*

$$\big[\nabla_4^l, \nabla\big]\phi = \sum_{i+j+k=l-1} \nabla_4^i \chi^{k+1} \cdot \nabla \nabla_4^j \phi + \sum_{i+j+k=l-1} \nabla \nabla_4^i \chi^{k+1} \cdot \nabla_4^j \phi,$$

$$\big[\nabla_3^l, \nabla\big]\phi = \sum_{i+j+k=l-1} \nabla_3^i \underline{\chi}^{k+1} \cdot \nabla \nabla_3^j \phi + \sum_{i+j+k=l-1} \nabla \nabla_3^i \underline{\chi}^{k+1} \cdot \nabla_3^j \phi,$$

where $\nabla_4^i \chi^{k+1}$ is a schematic notation for all possible ways of distributing i derivatives in the e_4 direction over a product of $k+1$ χ terms, and similarly in the e_3 direction.

Proof. As before, we have that $\big[\nabla_4, \nabla\big]\phi = \nabla \chi \cdot \phi + \chi \cdot \nabla \phi$ and we conclude by induction. □

We recall the notation for differences of the Ricci coefficients and their Minkowski values, as in the introduction:

$$\psi^* = \psi - \psi_{\text{Minkowski}}.$$

Similarly, we recall the notation for certain "good" curvature components:

$$\Psi^G \in \{\nu, \tau, R, \underline{\nu}, \underline{\alpha}\},\ \underline{\Psi}^G \in \{\alpha, \nu, \tau, R, \underline{\nu}\}.$$

Our terminology is motivated by the fact that the good curvature components Ψ^G extend continuously to $\{v = 0\}$ up to order $\frac{n-4}{2}$, whereas $\nabla_4^{\frac{n-4}{2}}\alpha$ is mildly singular at $\{v = 0\}$. A similar statement holds for $\underline{\Psi}^G$ and $\underline{\alpha}$ at $\{u = 0\}$.

We conclude this section by introducing the schematic error term notation that will play an fundamental role in keeping track of nonlinear error terms throughout the work. We adapt the notation of [RSR18, Definition 5.15] to our setting by defining:

Definition 2.4. *For any $m + l \leq p$, we introduce the schematic notation:*

$$\mathcal{F}_{mlp}(\Psi) = \sum_{\substack{i+j+k\leq p\\ i\leq l, k\leq m}} \nabla^k\nabla_4^i\big(\psi^{j+1}\Psi\big),\ \mathcal{F}_{mlp}(\Psi^G) = \sum_{\substack{i+j+k\leq p\\ i\leq l, k\leq m}} \nabla^k\nabla_4^i\big(\psi^{j+1}\Psi^G\big)$$

$$\underline{\mathcal{F}}_{mlp}(\Psi) = \sum_{\substack{i+j+k\leq p\\ i\leq l, k\leq m}} \nabla^k\nabla_3^i\big(\psi^{j+1}\Psi\big),\ \underline{\mathcal{F}}_{mlp}(\underline{\Psi}^G) = \sum_{\substack{i+j+k\leq p\\ i\leq l, k\leq m}} \nabla^k\nabla_3^i\big(\psi^{j+1}\underline{\Psi}^G\big),$$

where the terms $\nabla^k\nabla_4^i\big(\psi^{j+1}\Psi\big)$ denote the sum of all the possible products obtained when distributing the $\nabla^k\nabla_4^i$ derivatives. We also define $\mathcal{F}'(\Psi)$ and $\underline{\mathcal{F}}'(\Psi)$ as above, in the case when at least one of the Ricci coefficients is ψ^*.

We identify the top order term in $\mathcal{F}_{mlp}(\Psi)$ as being $\psi\nabla^m\nabla_4^l\Psi$. We write:

$$\mathcal{F}_{mlp}(\Psi) = \psi\nabla^m\nabla_4^l\Psi + \mathcal{F}_{mlp}^{lot}(\Psi),$$

with the understanding that when we expand $\mathcal{F}_{mlp}^{lot}(\Psi)$ using the product rule it does not contain any top order terms. Similarly, we also define $\mathcal{F}_{mlp}^{lot}(\Psi^G)$, $\underline{\mathcal{F}}_{mlp}^{lot}(\Psi)$, and $\underline{\mathcal{F}}_{mlp}^{lot}(\underline{\Psi}^G)$.

Using the above commutation formulas in Lemmas 2.4 and 2.5, we can prove by induction that:

Lemma 2.6. *The error terms $\mathcal{F}_{mlp}$ and $\underline{\mathcal{F}}_{mlp}$ satisfy:*

$$\nabla^i\nabla_4^j\mathcal{F}_{mlp}(\Psi) = \mathcal{F}_{(m+i)(l+j)(p+i+j)}(\Psi),\ \nabla^i\nabla_3^j\underline{\mathcal{F}}_{mlp}(\Psi) = \underline{\mathcal{F}}_{(m+i)(l+j)(p+i+j)}(\Psi).$$

Similar results hold for $\mathcal{F}_{mlp}(\Psi^G)$, $\underline{\mathcal{F}}_{mlp}(\underline{\Psi}^G)$, $\mathcal{F}'_{mlp}(\Psi)$ *and* $\underline{\mathcal{F}}'_{mlp}(\Psi)$.

Finally, we restate Lemmas 2.4 and 2.5 using the schematic error term notation:

Lemma 2.7. *For any horizontal tensor* ϕ, *we have the commutation formulas:*

$$\big[\nabla^m, \nabla_4\big]\phi = \mathcal{F}_{(m)(0)(m)}(\phi), \ \big[\nabla^m, \nabla_4^2\big]\phi = \mathcal{F}_{(m)(1)(m+1)}(\phi) \tag{2.31}$$

$$\big[\nabla_4^l, \nabla\big]\phi = \mathcal{F}_{(1)(l-1)(l)}(\phi), \ \big[\nabla_4^l, \Delta\big]\phi = \mathcal{F}_{(2)(l-1)(l+1)}(\phi). \tag{2.32}$$

The same result holds if we replace e_4 *by* e_3 *and* $\mathcal{F}$ *by* $\underline{\mathcal{F}}$.

Proof. The first three formulas follows directly from Lemmas 2.4 and 2.5 using the notation introduced above. Applying the third formula, we also get the last formula. □

3 Existence and Uniqueness of Scattering States

The main result of this section is the proof of the first statement of Theorem 1.3, establishing global existence and quantitative estimates for the solution in the region $\{u < 0, \ v > 0\}$, given small scattering data at $\{v = 0\}$. In the original $(n+1)$-dimensional formulation, this represents the proof of the first statement of Theorem 1.1, by showing the stability of de Sitter space with scattering data at $\mathcal{I}^-$. This section is based on [Cic24, Section 3]. We prove the following result:

Theorem 3.1. *For any* $N > 0$ *large enough there exist* $\epsilon_0 > 0$ *small enough, and a universal constant* $c_0 > 0$, *such that for any* $\epsilon \leq \epsilon_0$, *if the straight initial data* $\big(\slashed{g}_0, \check{h}\big)$ *satisfies the smallness assumption:*

$$\sum_{i=0}^{N} \big\|\mathcal{L}_\theta^i \slashed{g}_0^*\big\|_{L^2(S^n)} + \big\|h\big\|_{H^N(S^n)} < \epsilon, \tag{3.1}$$

then the corresponding straight self-similar vacuum spacetime $\big(\mathcal{M}, g\big)$ *with data at* $\{u = -1, \ v = 0\}$ *given by* $\big(\slashed{g}_0, \check{h}\big)$ *exists globally on* $\{u < 0, v > 0\}$, *extends to* $\{v = 0\}$ *as a regular solution, and satisfies quantitative estimates with regularity* $N' = N - c_0 n$, *as made precise in Propositions 3.1, 3.2, and 3.8 (with* $N_1, N_2, N_3 = N'$*).*

Moreover, there exists a small constant $\underline{v} > 0$, *with* $\epsilon \ll \underline{v} \ll 1$, *such that for small Cauchy initial data on a spacelike hypersurface* $\{v = -cu\}$ *with* $\underline{v} \leq c \leq \underline{v}^{-1}$, *as made precise in Remark 3.8, the corresponding straight self-similar vacuum spacetime* $\big(\mathcal{M}, g\big)$ *exists globally on* $\{u < 0, v >$

$0\}$ *and satisfies quantitative estimates with regularity* $N' = N - c_0 n$*, as made precise in Proposition 3.2 in region II, Proposition 3.8 in region III, and analogously to Proposition 3.8 in region I (with* $N_2, N_3 = N'$*).*

Remark 3.1. *In (3.1) we denote by* $\mathcal{L}_\theta^i$ *all the possible combinations of* i *Lie angular derivatives in a coordinate patch, and we sum over a family of coordinate patches that covers all of* S^n*.*

Remark 3.2. *We notice that specifying initial data* $(\not{g}_0, \check{h})$ *is equivalent to specifying initial data* $(\not{g}_0, h)$*, where* h *is the term in the expansion (1.21) of* α*. This follows since* $h = \check{h} + \big(1 + \frac{1}{2} + \ldots + \frac{2}{n}\big)\mathcal{O}$*, and* $\mathcal{O}$ *can be computed using* n *derivatives of* $\not{g}_0$*. Unless explicitly noted, we shall refer to* $(\not{g}_0, h)$ *as initial data for the rest of the work.*

Remark 3.3. *We recall according to the introduction that the assumption on the asymptotic data of* ϵ*-smallness of order* M *implies (1.19). This also implies (3.1) with* $N = M$ *and replacing* ϵ *by* $C\epsilon$ *(the proof follows from (3.48)).*

Remark 3.4. *The proof of Theorem 3.1 follows from Propositions 3.1, 3.2, and 3.8. At the the end of this section we also prove Theorem 3.2, establishing propagation of regularity. The proof of the first statement of Theorem 1.3 then follows from Theorem 3.1, Theorem 3.2, and the above remarks.*

As explained in Section 1.3.1 of the introduction, the proof follows the steps of [RSR23]. For small $\underline{v} > 0$ with $\epsilon \ll \underline{v} \ll 1$, we consider the following regions of the spacetime:

$$I = \left\{0 \leq \frac{v}{|u|} \leq \underline{v}\right\},\ II = \left\{\underline{v} \leq \frac{v}{|u|} \leq \underline{v}^{-1}\right\},\ III = \left\{0 < \frac{|u|}{v} \leq \underline{v}\right\}.$$

We also set $\delta > 0$ to be a small constant, and consider $\epsilon > 0$ small enough such that $C(\underline{v}) \leq \epsilon^{-\delta}$, where $C(\underline{v})$ is a constant determined in the proof.

Notation. We make the convention that in Section 3 and Section 4 we write $A \lesssim B$ for any quantities $A, B > 0$, if there is a constant $C > 0$ depending only on N such that $A \leq CB$.

Absolute value convention. We make the convention for the rest of the thesis that for any horizontal tensor ϕ defined on $S_{u,v}$, its absolute value $|\phi|$ is defined with respect to the metric $\not{g}(u, v)$ induced on $S_{u,v}$.

Integration convention. We make the convention that in Section 3 and Section 4 the volume forms used are $d\mathring{V}ol$, $d\mathring{V}ol\, du$, $d\mathring{V}ol\, dv$, and $d\mathring{V}ol\, dudv$, as needed in each context, where $d\mathring{V}ol$ represents the volume form with respect to the round metric $\left(\not{g}_0\right)_{\text{Minkowski}} = \frac{1}{4}\not{g}_{S^n}$.

Error terms convention. We make the convention for the rest of the thesis that if an index denoting the order of some derivative is negative, then that term is empty. For example $\mathcal{F}_{(m)(-1)(p)} := 0$, and so on.

3.1 Regions I and II

Given straight initial data $\left(\not{g}_0, h\right)$, [RSR18, Theorem 1.1] implies the existence of a unique regular straight self-similar vacuum solution in region I. Under the additional smallness assumption (3.1) for the data, one can repeat the argument of [RSR18, Theorem 1.1] with a few modifications to prove:

Proposition 3.1. *If the straight initial data* $\left(\not{g}_0, h\right)$ *satisfies the smallness assumption (3.1), then there exists a regular straight self-similar vacuum solution in region I with the given initial data. Setting* $N_1 = N - \lfloor c_0/4 \rfloor n$*, we have the bounds in region I for all* $0 \le i \le N_1$, $0 \le j \le \frac{n-4}{2}$, $0 \le k \le \frac{n}{2}$*:*

$$\left\| \nabla^i \nabla_4^j \nabla_3^k \Big(\alpha - v^{\frac{n-4}{2}} u^{\frac{4-n}{2}} \log(-v/u) \mathcal{O} / \big((n-4)/2\big)! \Big) \right\|_{L^\infty(S_{u,v})} \lesssim \epsilon |u|^{-2-i-j-k} \tag{3.2}$$

$$\left\| \nabla^i \nabla_4^j \nabla_3^k \Psi^G \right\|_{L^\infty(S_{u,v})} \lesssim \epsilon |u|^{-2-i-j-k} \tag{3.3}$$

$$\left\| \nabla^i \nabla_4^j \nabla_3^k \psi^* \right\|_{L^\infty(S_{u,v})} \lesssim \epsilon |u|^{-1-i-j-k} \tag{3.4}$$

$$\left\| \nabla^i \nabla_4^{\frac{n-2}{2}} \nabla_3^k \psi^* \right\|_{L^\infty(S_{u,v})} \lesssim \epsilon \cdot \log\left(\frac{|u|}{v} \right) \cdot |u|^{-1-i-\frac{n-2}{2}-k} \tag{3.5}$$

$$\left\| \mathcal{L}_\theta^i \not{g}^* \right\|_{L^\infty(S_{u,v})} \lesssim \epsilon |u|^{-i}. \tag{3.6}$$

Remark 3.5. *We explain briefly how one can adapt the proof of [RSR18, Theorem 1.1] to obtain Proposition 3.1. Firstly, we point out that [RSR18, Proposition 7.3] establishes the estimates* (3.2)-(3.6) *without the* ϵ *factor. Moreover, one can check that the smallness assumption (3.1) implies that the initial data norms used in the proof of [RSR18, Theorem 1.1] are small in terms of* ϵ*. Therefore, one can repeat the proof of [RSR18, Theorem 1.1] and also propagate the smallness of the initial data. The same argument as in [RSR18, Proposition 7.3] then implies Proposition 3.1.*

Remark 3.6. *The proof of [RSR18, Theorem 1.1] implies the need to prove estimates for the solution with angular regularity at most* N_1*, which is sufficiently small compared to the regularity of the initial data* N*. In this section we do not attempt to optimize the universal constant* c_0*, and for our purposes it suffices to take* $c_0 = 100$*.*

To obtain estimates in region II, one can adapt the argument of [RSR23, Section 7] to the case of $n \geq 4$ and obtain the following bounds:

Proposition 3.2. *The solution of Proposition 3.1 can be extended uniquely as a regular straight self-similar vacuum solution in region II. Setting* $N_2 = N - 2\lfloor c_0/4 \rfloor n$*, we have the bounds in region II for all* $0 \leq i \leq N_2$*, and all* $0 \leq j, k \leq \frac{n-4}{2}$*:*

$$\left\| \nabla^i \nabla_4^j \nabla_3^k \Psi \right\|_{L^\infty(S_{u,v})} \lesssim \epsilon^{1-\delta} |v|^{-2-i-j-k} \tag{3.7}$$

$$\left\| \nabla^i \nabla_4^j \nabla_3^k \psi^* \right\|_{L^\infty(S_{u,v})} \lesssim \epsilon^{1-\delta} |v|^{-1-i-j-k} \tag{3.8}$$

$$\left\| \nabla^i \nabla_4^{\frac{n-2}{2}} \nabla_3^k \psi^* \right\|_{L^\infty(S_{u,v})} \lesssim \epsilon^{1-\delta} |v|^{-1-i-\frac{n-2}{2}-k} \tag{3.9}$$

$$\left\| \mathcal{L}_\theta^i \slashed{g}^* \right\|_{L^\infty(S_{u,v})} \lesssim \epsilon^{1-\delta} |v|^{-i}. \tag{3.10}$$

Remark 3.7. *The argument of [RSR23, Section 7] is carried out in the case of* $n = 2$*. In order to prove Proposition 3.2, one must adapt this argument to the case of* $n \geq 4$ *even, by making the standard changes specific to working in higher dimensions. According to Definition 2.3, for* $(n+2)$*-dimensional straight self-similar metrics the double null unknowns solve the metric equations* (2.7)-(2.8)*, the system of null structure equations in Proposition 2.4, the Bianchi equations in Proposition 2.6, and the system of constraint equations in Propositions 2.5 and 2.7. The structure of the equations satisfied by the double null unknowns is the same as in the case of* $n = 2$*, which allows us to adapt the proof of [RSR23, Section 7] and establish Proposition 3.2.*

Remark 3.8. *With the same argument as in Remark 3.7, the conclusion of Proposition 3.2 also holds in the case of small Cauchy initial data on a spacelike hypersurface* $\{v = -cu\}$*, with* $\underline{v} \leq c \leq \underline{v}^{-1}$*. We consider a set of double null unknowns* $\slashed{g}, \psi, \Psi$ *which satisfy the constraint equations in Propositions 2.5, 2.7, and the following smallness conditions for all* $0 \leq i \leq N_1$*, and all* $0 \leq j, k \leq \frac{n-2}{2}$*:*

$$\left\| \nabla^i \nabla_4^j \nabla_3^k \Psi \right\|_{L^\infty(S_{-cv,v})} \lesssim \epsilon |v|^{-2-i-j-k} \tag{3.11}$$

$$\left\|\nabla^i\nabla_4^j\nabla_3^k\psi^*\right\|_{L^\infty(S_{-cv,v})} \lesssim \epsilon|v|^{-1-i-j-k} \tag{3.12}$$

$$\left\|\mathcal{L}_\theta^i \slashed{g}^*\right\|_{L^\infty(S_{-cv,v})} \lesssim \epsilon|v|^{-i}. \tag{3.13}$$

This is the precise notion in which the spacetime $\big(\mathcal{M}, g\big)$ *determined by the initial data is close to Minkowski space on the spacelike hypersurface* $\{v = -cu\}$, *as referred to in Theorem 1.3 and Theorem 3.1.*

3.2 Region III

We prove existence of the solution and self-similar bounds in region III using a bootstrap argument. We first introduce the norms used in the bootstrap argument in Section 3.2.1. We then state the bootstrap assumptions in Section 3.2.2. Under the bootstrap assumptions, we prove improved estimates in Sections 3.2.3-3.2.6. In Section 3.2.7, we use the results of Propositions 3.3-3.6 to complete the bootstrap argument in Proposition 3.7. As a result, we obtain the main statement on the existence of the solution satisfying quantitative bounds in Proposition 3.8, which together with Propositions 3.1 and 3.2 completes the proof of Theorem 3.1. Finally, we prove propagation of regularity in Theorem 3.2.

3.2.1 Norms

We define the norms used in the bootstrap argument. We set $N_3 = N - 3\lfloor c_0/4\rfloor n$ and consider $p > 0$ to be a small constant. We introduce the following sets of indices for high and low regularity norms:

$$H = \left\{(m,l):\ 0 \le l \le \frac{n-4}{2},\ 0 \le m \le N_3 + \frac{n-4}{2} - l\right\}$$

$$L = \left\{(m,l):\ 0 \le l \le \frac{n-4}{2},\ 0 \le m \le N_3 + \frac{n-6}{2} - l\right\}.$$

We also define the characteristic triangles:

$$P_{\widetilde{u},\widetilde{v}} = \left\{(u,v):\ -\underline{v}v \le u \le \widetilde{u},\ -\underline{v}^{-1}u \le v \le \widetilde{v}\right\}. \tag{3.14}$$

We define the high regularity curvature norms in region III with $v < \underline{v}$, where $w_{ml}(u,v) = v^{\frac{3}{2}+m+l-p}|u|^{p-q}$ for some constants $0 < q \ll p \ll 1$, and $(m,l) \in H$. We recall our above

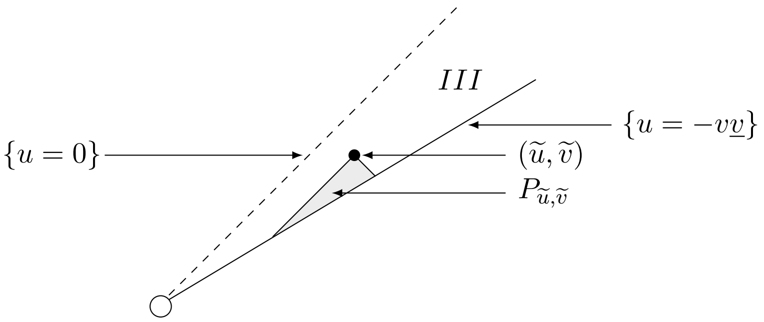

Figure 2.1: The characteristic triangle $P_{\widetilde{u},\widetilde{v}}$

conventions for absolute value and integration.

$$\begin{aligned}\left\|\underline{\alpha}\right\|^2_{\mathcal{C}_{m,l}}(u,v) =& |u|^{2q}\int_{-v\underline{v}}^{u}\int_{S^n} w_{ml}^2\left|\nabla^m\nabla_3^l\underline{\alpha}\right|^2 d\mathring{V}ol\,d\hat{u}\\ &+|u|^{2q}\int_{-v\underline{v}}^{u}\int_{-\frac{\hat{u}}{\underline{v}}}^{v}\int_{S^n}\frac{w_{ml}^2}{\hat{v}}\left|\nabla^m\nabla_3^l\underline{\alpha}\right|^2 d\mathring{V}ol\,d\hat{v}\,d\hat{u}.\end{aligned}$$

For any $\underline{\Psi}^G\neq\alpha$, we define:

$$\begin{aligned}&\left\|\underline{\Psi}^G\right\|^2_{\mathcal{C}_{m,l}}(u,v) = |u|^{2q}\int_{-v\underline{v}}^{u}\int_{S^n} w_{ml}^2\left|\nabla^m\nabla_3^l\underline{\Psi}^G\right|^2 d\mathring{V}ol\,d\hat{u}\\ &+|u|^{2q}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n} w_{ml}^2\left|\nabla^m\nabla_3^l\underline{\Psi}^G\right|^2 d\mathring{V}ol\,d\hat{v}+|u|^{2q}\int_{-v\underline{v}}^{u}\int_{-\frac{\hat{u}}{\underline{v}}}^{v}\int_{S^n}\frac{w_{ml}^2}{|\hat{u}|}\left|\nabla^m\nabla_3^l\underline{\Psi}^G\right|^2 d\mathring{V}ol\,d\hat{v}\,d\hat{u}.\end{aligned}$$

For α, we define:

$$\begin{aligned}\left\|\alpha\right\|^2_{\mathcal{C}_{m,l}}(u,v) =& |u|^{2q}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n} w_{ml}^2\left|\nabla^m\nabla_3^l\alpha\right|^2 d\mathring{V}ol\,d\hat{v}\\ &+|u|^{2q}\int_{-v\underline{v}}^{u}\int_{-\frac{\hat{u}}{\underline{v}}}^{v}\int_{S^n}\frac{w_{ml}^2}{|\hat{u}|}\left|\nabla^m\nabla_3^l\alpha\right|^2 d\mathring{V}ol\,d\hat{v}\,d\hat{u}.\end{aligned}$$

We define the total high regularity curvature norm as:

$$\mathcal{C}_{\widetilde{u},\widetilde{v}} = \sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}}\sum_{(m,l)\in H}\Bigg(\left\|\underline{\alpha}\right\|^2_{\mathcal{C}_{m,l}}(u,v)+\left\|\underline{\Psi}^G\right\|^2_{\mathcal{C}_{m,l}}(u,v)\Bigg). \tag{3.15}$$

We define the low regularity curvature norms in region III with $v<\underline{v}$ for $(m,l)\in L$:

$$\begin{aligned}\left\|\underline{\Psi}^G\right\|^2_{\mathcal{L}_{m,l}}(u,v) &= \int_{S^n} v^{4+2m+2l}\left|\nabla^m\nabla_3^l\underline{\Psi}^G\right|^2 d\mathring{V}ol\\ \left\|\underline{\alpha}\right\|^2_{\mathcal{L}_{m,l}}(u,v) &= \int_{S^n} v^{4+2m+2l-2p}|u|^{2p}\left|\nabla^m\nabla_3^l\underline{\alpha}\right|^2 d\mathring{V}ol.\end{aligned}$$

We define the total low regularity curvature norm as:

$$\mathcal{L}_{\widetilde{u},\widetilde{v}} = \sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}} \sum_{(m,l)\in L} \Big(\big\|\underline{\alpha}\big\|^2_{\mathcal{L}_{m,l}}(u,v) + \big\|\underline{\Psi}^G\big\|^2_{\mathcal{L}_{m,l}}(u,v) \Big). \tag{3.16}$$

Remark 3.9. *The curvature components represent the most important double null unknowns for our argument. At top order, we use energy estimates to bound the null flux terms and the spacetime bulk terms in the total high regularity curvature norm* (3.15). *When dealing with nonlinear error terms, it is practical to also keep track of the total low regularity curvature norm* (3.16)*, which in particular implies pointwise bounds for the curvature components via Sobolev inequalities.*

We define the norms for Ricci coefficients in region III with $v < \underline{v}$ for $(m,l) \in H$:

$$\big\|\psi\big\|^2_{\mathcal{R}_{m,l}}(u,v) = \int_{S^n} v^{2+2m+2l} \big|\nabla^m \nabla_3^l \psi^*\big|^2 d\mathring{Vol}.$$

We define the total Ricci coefficients norm as:

$$\mathcal{R}_{\widetilde{u},\widetilde{v}} = \sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}} \sum_{(m,l)\in H} \big\|\psi\big\|^2_{\mathcal{R}_{m,l}}(u,v). \tag{3.17}$$

Finally, we define the norms for the metric coefficients for $(m,0) \in H$:

$$\big\|\slashed{g}\big\|^2_{\mathcal{M}_{m,0}}(u,v) = \int_{S^n} v^{2m} \big|\mathcal{L}_\theta^m \slashed{g}^*\big|^2 d\mathring{Vol}.$$

We define the total metric coefficients norm as:

$$\mathcal{M}_{\widetilde{u},\widetilde{v}} = \sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}} \sum_{(m,0)\in H} \big\|\slashed{g}\big\|^2_{\mathcal{M}_{m,0}}(u,v). \tag{3.18}$$

3.2.2 Bootstrap Assumptions

Let $\big(\mathcal{M}, g\big)$ be a spacetime obtained in Proposition 3.2, which exists in the characteristic triangle $P_{\widetilde{u},\widetilde{v}}$ contained in the region III with $\widetilde{v} \leq \underline{v}$. We denote $\epsilon' = \epsilon^{1-2\delta}$. For a sufficiently large constant $A > 0$, depending on N and n, we make the following bootstrap assumption:

$$\mathcal{C}_{\widetilde{u},\widetilde{v}} + \mathcal{L}_{\widetilde{u},\widetilde{v}} + \mathcal{R}_{\widetilde{u},\widetilde{v}} + \mathcal{M}_{\widetilde{u},\widetilde{v}} \leq A\epsilon'^2. \tag{3.19}$$

Our goal is to prove in Proposition 3.7 that we can improve the bootstrap assumption (3.19).

We remark that the bootstrap assumption holds initially on $\{(u,v): \ u = -\underline{v}v, \ 0 < v \leq \underline{v}\}$. For all $0 < v \leq \underline{v}$, we have by Proposition 3.2 and the definitions (3.15)-(3.18):

$$\mathcal{C}_{-\underline{v}v,v} + \mathcal{L}_{-\underline{v}v,v} + \mathcal{R}_{-\underline{v}v,v} + \mathcal{M}_{-\underline{v}v,v} \lesssim \epsilon'^2. \tag{3.20}$$

We note some consequences of the bootstrap assumption (3.19). We have for all $(u,v) \in P_{\widetilde{u},\widetilde{v}}$ and $(m,l) \in L$:

$$\int_{S^n} v^{2+2m+2l} \left|\nabla^m \nabla_3^l \psi\right|^2 d\mathring{Vol} \lesssim 1, \tag{3.21}$$

$$\int_{S^n} v^{2m} \left|\mathcal{L}_\theta^m \not{g}\right|^2 d\mathring{Vol} \lesssim 1. \tag{3.22}$$

Another essential consequence of the bootstrap assumption (3.19) is that we can use Sobolev inequalities, which play a key role in bounding nonlinear error terms. We use the definition [RSR23, Definition 6.11] for the weighted $\widetilde{H}^m(S_{u,v})$ Sobolev spaces:

$$\left\|\phi\right\|_{\widetilde{H}^m(S_{u,v})} = \sum_{i=0}^{m} (v-u)^i \left(\int_{S^n} \left|\nabla^i \phi\right|^2 d\mathring{Vol} \right)^{\frac{1}{2}}. \tag{3.23}$$

According to [RSR23, Lemma 6.5], the bootstrap assumption (3.19) implies that for any $(m,0) \in H$ we have the Sobolev inequalities:

$$\left\|\phi\right\|_{L^\infty(S_{u,v})} \lesssim \left\|\phi\right\|_{\widetilde{H}^n(S_{u,v})} \tag{3.24}$$

$$\left\|\phi \cdot \psi\right\|_{\widetilde{H}^m(S_{u,v})} \lesssim \left\|\phi\right\|_{\widetilde{H}^m(S_{u,v})} \left\|\psi\right\|_{\widetilde{H}^m(S_{u,v})}. \tag{3.25}$$

3.2.3 Estimates for High Regularity Curvature Components

In this section, we show improved bounds on the high regularity curvature components (3.15) by proving:

Proposition 3.3. *Assuming that the bootstrap assumption* (3.19) *holds, there exists a constant* $C \ll A$, *such that we have the improved estimate for the high regularity curvature components* (3.15)*:*

$$\mathcal{C}_{\widetilde{u},\widetilde{v}} \leq C \epsilon'^2. \tag{3.26}$$

Proof. Step 1. The Bianchi pair $(\underline{\nu}, \underline{\alpha})$. We first rewrite the Bianchi equations (2.28)-(2.29) in a form that is suitable for region III, noting that the error term notation in Definition 2.2 is adapted to region I. By analogy to equations (2.22)-(2.23), the equations (2.28)-(2.29) can be written schematically as:

$$\nabla_3 \underline{\nu}_{ABC} = -2\nabla_{[A} \underline{\alpha}_{B]C} + \sum_{s_1+s_2=5/2} \psi_{s_1} \Psi_{s_2},$$

$$\nabla_4\underline{\alpha}_{AB} + \frac{1}{2}\mathrm{tr}\chi\underline{\alpha}_{AB} = -\nabla^C\underline{\nu}_{C(AB)} + \sum_{s_1+s_2=2,\ s_1\neq 0}\psi_{s_1}\Psi_{s_2}.$$

Using the signature properties in Lemma 2.3, we can rewrite the equations for $(\underline{\nu},\underline{\alpha})$ as:

$$\nabla_3\underline{\nu}_{ABC} = -2\nabla_{[A}\underline{\alpha}_{B]C} + \psi\underline{\Psi}^G, \tag{3.27}$$

$$\nabla_4\underline{\alpha}_{AB} + \frac{1}{2}\mathrm{tr}\chi\underline{\alpha}_{AB} = -\nabla^C\underline{\nu}_{C(AB)} + \psi\underline{\Psi}^G. \tag{3.28}$$

Step 1a. The system of commuted equations. For any $(m,l)\in H$, we commute equations (3.27)-(3.28) with ∇_3^l using Lemma 2.7:

$$\nabla_3\nabla_3^l\underline{\nu}_{ABC} = -2\nabla_{[A}\nabla_3^l\underline{\alpha}_{B]C} + \underline{\mathcal{F}}_{(1)(l-1)(l)}(\Psi) + \underline{\mathcal{F}}_{(0)(l)(l)}(\underline{\Psi}^G),$$

$$\begin{aligned}\nabla_4\nabla_3^l\underline{\alpha}_{AB} + \frac{1}{2}\mathrm{tr}\chi\nabla_3^l\underline{\alpha}_{AB} =& -\nabla^C\nabla_3^l\underline{\nu}_{C(AB)} + \underline{\mathcal{F}}'_{(0)(l)(l)}(\Psi) + \\ &+ \underline{\mathcal{F}}_{(1)(l-1)(l)}(\underline{\Psi}^G) + \underline{\mathcal{F}}_{(0)(l)(l)}(\underline{\Psi}^G) + \sum_{i=0}^{l-1}\nabla_3^{l-i}\mathrm{tr}\chi\nabla^m\nabla_3^i\underline{\alpha},\end{aligned}$$

where we used the fact that $\nabla\psi = \nabla\psi^*$. We further commute the equations with ∇^m as well, using Lemmas 2.4, 2.6, and 2.7:

$$\begin{aligned}\nabla_3\nabla^m\nabla_3^l\underline{\nu}_{ABC} =& -2\nabla_{[A}\nabla^m\nabla_3^l\underline{\alpha}_{B]C} + \underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\Psi) + \\ &+ \underline{\mathcal{F}}_{(m)(l)(m+l)}(\underline{\Psi}^G) + \sum_{i+2j=m-1}\nabla^i\big(Ri\not{e}m^{j+1}\nabla_3^l\underline{\alpha}\big),\end{aligned}$$

$$\nabla_4\nabla^m\nabla_3^l\underline{\alpha}_{AB} + \left(\frac{1}{2}+\frac{m}{n}\right)\mathrm{tr}\chi\nabla^m\nabla_3^l\underline{\alpha}_{AB} = -\nabla^C\nabla^m\nabla_3^l\underline{\nu}_{C(AB)} + \underline{\mathcal{F}}'_{(m)(l)(m+l)}(\Psi) +$$

$$+\underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\underline{\Psi}^G) + \underline{\mathcal{F}}_{(m)(l)(m+l)}(\underline{\Psi}^G) + \sum_{i+2j=m-1}\nabla^i\big(Ri\not{e}m^{j+1}\nabla_3^l\underline{\nu}\big) + \sum_{i=0}^{l-1}\nabla_3^{l-i}\mathrm{tr}\chi\nabla^m\nabla_3^i\underline{\alpha}.$$

We also recall that $\mathrm{tr}\chi = \mathrm{tr}\chi^* + n/(v-u)$ and $\mathrm{tr}\underline{\chi} = \mathrm{tr}\underline{\chi}^* - n/(v-u)$. The last term in the second equation above can be written as:

$$\sum_{i=0}^{l-1}\nabla_3^{l-i}\mathrm{tr}\chi\nabla^m\nabla_3^i\underline{\alpha} = \underline{\mathcal{F}}'_{(m)(l)(m+l)}(\Psi) + O\bigg(\sum_{i=0}^{l-1}\frac{\big|\nabla^m\nabla_3^i\underline{\alpha}\big|}{(v-u)^{l-i+1}}\bigg).$$

We define $w_{ml} = v^{\frac{3}{2}+m+l-p}|u|^{p-q}$ and derive equations for $w_{ml}\nabla^m\nabla_3^l\underline{\nu}_{ABC}$ and $w_{ml}\nabla^m\nabla_3^l\underline{\alpha}_{AB}$. For the rest of the section we refer to this procedure as "conjugating the equations with w_{ml}".

$$\nabla_3 w_{ml}\nabla^m\nabla_3^l\underline{\nu}_{ABC} + \frac{p-q}{|u|}w_{ml}\nabla^m\nabla_3^l\underline{\nu}_{ABC} = -2\nabla_{[A}w_{ml}\nabla^m\nabla_3^l\underline{\alpha}_{B]C} + w_{ml}Err_{ml}^{\underline{\nu}}, \tag{3.29}$$

$$\nabla_4 w_{ml}\nabla^m\nabla_3^l\underline{\alpha}_{AB} + \Big(\frac{n-3}{2} - l + p\Big)\frac{w_{ml}}{v}\nabla^m\nabla_3^l\underline{\alpha}_{AB} = -\nabla^C w_{ml}\nabla^m\nabla_3^l\underline{\nu}_{C(AB)} + \tag{3.30}$$
$$+ w_{ml}Err^{\underline{\alpha}}_{ml} + O\Big(\underline{v}\cdot\frac{w_{ml}}{v}\cdot\big|\nabla^m\nabla_3^l\underline{\alpha}\big|\Big),$$

where we have the error terms:

$$Err^{\underline{\nu}}_{ml} = \underline{\mathcal{F}}_{(m)(l)(m+l)}(\Psi) + \underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\Psi) + \sum_{i+2j=m-1}\nabla^i\big(Ri\not{e}m^{j+1}\nabla_3^l\underline{\alpha}\big), \tag{3.31}$$

$$Err^{\underline{\alpha}}_{ml} = \underline{\mathcal{F}}_{(m)(l)(m+l)}(\underline{\Psi}^G) + \underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\underline{\Psi}^G) + \underline{\mathcal{F}}'_{(m)(l)(m+l)}(\Psi) \tag{3.32}$$
$$+ \sum_{i+2j=m-1}\nabla^i\big(Ri\not{e}m^{j+1}\nabla_3^l\underline{\nu}\big) + O\Big(\sum_{i=0}^{l-1}\frac{\big|\nabla^m\nabla_3^i\underline{\alpha}\big|}{(v-u)^{l-i+1}}\Big).$$

Step 1b. The energy estimates for $(\underline{\nu},\underline{\alpha})$. For simplicity, we denote $\mathcal{D} = w_{ml}\nabla^m\nabla_3^l$. We notice that integration by parts and the bootstrap assumption (3.19) give:

$$\int_{S^n}\Big(-\nabla_{[A}\mathcal{D}\underline{\alpha}_{B]C}\cdot\mathcal{D}\underline{\nu}^{ABC} - \mathcal{D}\underline{\alpha}^{AB}\cdot\nabla^C\mathcal{D}\underline{\nu}_{C(AB)}\Big)d\mathring{V}ol =$$
$$= \int_{S^n}\nabla_A\mathcal{D}\underline{\alpha}_{BC}\cdot\mathcal{D}\underline{\nu}^{A[BC]}d\mathring{V}ol + O\Big(\int_{S^n}\frac{1}{v}\big|\mathcal{D}\underline{\alpha}\big|\cdot\big|\mathcal{D}\underline{\nu}\big|d\mathring{V}ol\Big)$$
$$= O\Big(v^{-1}\int_{S^n}\big|\mathcal{D}\underline{\alpha}\big|\cdot\big|\mathcal{D}\underline{\nu}\big|d\mathring{V}ol\Big).$$

We use this identity in order to prove energy estimates for the Bianchi pair $(\underline{\nu},\underline{\alpha})$. We contract equation (3.29) with $\frac{1}{2}\mathcal{D}\underline{\nu}$, we contract equation (3.30) with $\mathcal{D}\underline{\alpha}$, and add the resulting equations. We integrate over $S_{u,v}$ and use the integration by parts identity to obtain:

$$\frac{1}{2}\partial_u\int_{S^n}\big|\mathcal{D}\underline{\nu}\big|^2 d\mathring{V}ol + \frac{p-q}{2|u|}\int_{S^n}\big|\mathcal{D}\underline{\nu}\big|^2 d\mathring{V}ol +$$
$$+\partial_v\int_{S^n}\big|\mathcal{D}\underline{\alpha}\big|^2 d\mathring{V}ol + \Big(\frac{n-3}{2} - l + p\Big)\frac{1}{v}\int_{S^n}\big|\mathcal{D}\underline{\alpha}\big|^2 d\mathring{V}ol =$$
$$= O\Big(\int_{S^n} v^{-1}\big|\mathcal{D}\underline{\alpha}\big|\cdot\big|\mathcal{D}\underline{\nu}\big| + w_{ml}\big|\mathcal{D}\underline{\nu}\big|\cdot\big|Err^{\underline{\nu}}_{ml}\big| + w_{ml}\big|\mathcal{D}\underline{\alpha}\big|\cdot\big|Err^{\underline{\alpha}}_{ml}\big| + \underline{v}v^{-1}\big|\mathcal{D}\underline{\alpha}\big|^2 d\mathring{V}ol\Big).$$

We integrate this identity over the characteristic triangle $P_{u,v}$ given by (3.14). We integrate by parts, then multiply everything by $|u|^{2q}$. The first and third terms above yield flux terms on $\{\hat{u} = u\}$, $\{\hat{v} = v\}$, and boundary terms on $\{u = -\underline{v}v\}$. The second and fourth terms above yield positive sign bulk terms. Finally, we use Cauchy-Schwarz, the positive sign bulk terms, and the fact that $|u|/v \leq \underline{v} \ll 1$ in order to absorb some of the error terms. We obtain the energy estimate in region III for $v < \underline{v}$:

$$|u|^{2q}\int_{-v\underline{v}}^{u}\int_{S^n} w_{ml}^2\big|\nabla^m\nabla_3^l\underline{\alpha}\big|^2 d\mathring{V}old\hat{u} + |u|^{2q}\int_{-v\underline{v}}^{u}\int_{-\frac{\hat{u}}{\underline{v}}}^{v}\int_{S^n}\frac{w_{ml}^2}{\hat{v}}\big|\nabla^m\nabla_3^l\underline{\alpha}\big|^2 d\mathring{V}old\hat{v}d\hat{u}+$$

$$+|u|^{2q}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n} w_{ml}^2\big|\nabla^m\nabla_3^l\underline{\nu}\big|^2 d\mathring{V}old\hat{v} + |u|^{2q}\int_{-v\underline{v}}^{u}\int_{-\frac{\hat{u}}{\underline{v}}}^{v}\int_{S^n}\frac{w_{ml}^2}{|\hat{u}|}\big|\nabla^m\nabla_3^l\underline{\nu}\big|^2 d\mathring{V}old\hat{v}d\hat{u} \lesssim$$

$$\lesssim \underline{v}^{2p-2q}|u|^{2q}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n}\hat{v}^{3+2m+2l-2q}\Big(\big|\nabla^m\nabla_3^l\underline{\alpha}\big|^2 + \big|\nabla^m\nabla_3^l\underline{\nu}\big|^2\Big)(-\underline{v}\hat{v},\hat{v})d\mathring{V}old\hat{v}+$$

$$+|u|^{2q}\int_{-v\underline{v}}^{u}\int_{-\frac{\hat{u}}{\underline{v}}}^{v}\int_{S^n}\Big(\hat{v}w_{ml}^2\big|Err_{ml}^{\underline{\alpha}}\big|^2 + |\hat{u}|w_{ml}^2\big|Err_{ml}^{\underline{\nu}}\big|^2\Big)d\mathring{V}old\hat{v}d\hat{u}.$$

We bound the data term on $\{u = -\underline{v}v\}$ using Proposition 3.2 by:

$$\underline{v}^{2p-2q}|u|^{2q}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n}\hat{v}^{3+2m+2l-2q}\Big(\big|\nabla^m\nabla_3^l\underline{\alpha}\big|^2 + \big|\nabla^m\nabla_3^l\underline{\nu}\big|^2\Big)(-\underline{v}\hat{v},\hat{v})$$

$$\lesssim \underline{v}^{2p-2q}|u|^{2q}\int_{-\frac{u}{\underline{v}}}^{v}(\epsilon')^2\hat{v}^{-1-2q}d\hat{v} \lesssim \underline{v}^{2p}(\epsilon')^2.$$

Here we notice the importance of the factor v^{-p} in the definition of w_{ml}, needed to avoid logarithmic degeneracy.

Step 1c. Bounding the error terms. We bound the error terms resulting from (3.31)-(3.32) one by one. We remark that:

$$\sum_{(m,l)\in H} w_{ml}^2\big|\underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\Psi)\big|^2 \lesssim \sum_{(m,l)\in H} w_{ml}^2\big|\underline{\mathcal{F}}_{(m)(l)(m+l)}(\Psi)\big|^2,$$

so bounding the first terms in $Err_{ml}^{\underline{\alpha}}$ and $Err_{ml}^{\underline{\nu}}$ will also imply control of the second terms once summing. For any $0 \leq i \leq l-1$, we have $(m,i)\in L$ and we get the bound:

$$|u|^{2q}\int_{-v\underline{v}}^{u}\int_{-\frac{\hat{u}}{\underline{v}}}^{v}\int_{S^n}\hat{v}w_{ml}^2\frac{\big|\nabla^m\nabla_3^i\underline{\alpha}\big|^2}{(\hat{v}-\hat{u})^{2l-2i+2}} \lesssim A\epsilon'^2\cdot|u|^{2q}\int_{-v\underline{v}}^{u}\int_{-\frac{\hat{u}}{\underline{v}}}^{v} w_{ml}^2|\hat{u}|^{-2p}\hat{v}^{-5-2m-2l+2p} \lesssim A\underline{v}\cdot\epsilon'^2.$$

Next, we have the bound using the Sobolev inequalities:

$$|u|^{2q}\int_{-v\underline{v}}^{u}\int_{-\frac{\hat{u}}{\underline{v}}}^{v}\int_{S^n}\hat{v}w_{ml}^2\big|\underline{\mathcal{F}}_{(m)(l)(m+l)}(\underline{\Psi}^G)\big|^2 \lesssim$$

$$\lesssim \sum_{\substack{i+j+k\leq m+l\\ i\leq l,k\leq m}}|u|^{2q}\int_{-v\underline{v}}^{u}\int_{-\frac{\hat{u}}{\underline{v}}}^{v}\int_{S^n}\hat{v}w_{ml}^2\big|\nabla^k\nabla_3^i\big(\psi^{j+1}\underline{\Psi}^G\big)\big|^2$$

$$\lesssim \sum_{\substack{|i|+j+|k|\leq m+l\\ |i|\leq l,|k|\leq m}}|u|^{2q}\int_{-v\underline{v}}^{u}\int_{-\frac{\hat{u}}{\underline{v}}}^{v}\int\frac{|\hat{u}|}{\hat{v}}\hat{v}^{3+2i_0+2k_0-2p}|\hat{u}|^{2p-2q-1}\Big|\nabla^{k_0}\nabla_3^{i_0}\underline{\Psi}^G\Big|\prod_{a=1}^{j+1}\hat{v}^{2+2i_a+2k_a}\Big|\nabla^{k_a}\nabla_3^{i_a}\psi\Big|$$

$$\lesssim |u|^{2q} \int_{-v\underline{v}}^{u} \int_{-\frac{\hat{u}}{\underline{v}}}^{v} \frac{|\hat{u}|}{\hat{v}} \sum_{(k_0,i_0)\in H} \hat{v}^{3+2i_0-2p} |\hat{u}|^{2p-2q-1} \left\| \nabla_3^{i_0} \underline{\Psi}^G \right\|_{\widetilde{H}^{k_0}}^2 \cdot \prod_{a=1}^{j+1} \sum_{(k_a,i_a)\in H} \hat{v}^{2+2i_a} \left\| \nabla_3^{i_a} \psi \right\|_{\widetilde{H}^{k_a}}^2$$

$$\lesssim \underline{v} \cdot \sum_{(k,i)\in H} \left\| \underline{\Psi}^G \right\|_{\mathcal{C}_{k,i}}^2 \lesssim \underline{v} \cdot A\epsilon'^2.$$

Following the same steps, we also have that:

$$|u|^{2q} \int_{-v\underline{v}}^{u} \int_{-\frac{\hat{u}}{\underline{v}}}^{v} \int_{S^n} |\hat{u}| w_{ml}^2 \left| \underline{\mathcal{F}}_{(m)(l)(m+l)}(\Psi) \right|^2 d\mathring{V}ol d\hat{v} d\hat{u} \lesssim$$

$$\lesssim |u|^{2q} \int_{-v\underline{v}}^{u} \int_{-\frac{\hat{u}}{\underline{v}}}^{v} \frac{|\hat{u}|}{\hat{v}} \sum_{(k_0,i_0)\in H} \hat{v}^{2+2i_0-2p} |\hat{u}|^{2p-2q} \left\| \nabla_3^{i_0} \Psi \right\|_{\widetilde{H}^{k_0}}^2 \cdot \prod_{a=1}^{j+1} \sum_{(k_a,i_a)\in H} \hat{v}^{2+2i_a} \left\| \nabla_3^{i_a} \psi \right\|_{\widetilde{H}^{k_a}}^2$$

$$\lesssim \underline{v} \cdot \sum_{(k,i)\in H} \left\| \Psi \right\|_{\mathcal{C}_{k,i}}^2 \lesssim \underline{v} \cdot A\epsilon'^2.$$

Similarly to the first error term, we have the estimate:

$$|u|^{2q} \int_{-v\underline{v}}^{u} \int_{-\frac{\hat{u}}{\underline{v}}}^{v} \int_{S^n} |\hat{v}| w_{ml}^2 \left| \underline{\mathcal{F}}'_{(m)(l)(m+l)}(\Psi) \right|^2 d\mathring{V}ol d\hat{v} d\hat{u} \lesssim$$

$$|u|^{2q} \iint_{P_{u,v}} \sum_{k_0,i_0} \hat{v}^{2+2i_0-2p} |\hat{u}|^{2p-2q} \left\| \nabla_3^{i_0} \Psi \right\|_{\widetilde{H}^{k_0}}^2 \sum_{k_1,i_1} \hat{v}^{2+2i_1} \left\| \nabla_3^{i_1} \psi^* \right\|_{\widetilde{H}^{k_1}}^2 \prod_{a=2}^{j+1} \sum_{k_a,i_a} \hat{v}^{2+2i_a} \left\| \nabla_3^{i_a} \psi \right\|_{\widetilde{H}^{k_a}}^2$$

$$\lesssim \mathcal{R}_{u,v} \cdot \sum_{(k,i)\in H} \left\| \Psi \right\|_{\mathcal{C}_{k,i}}^2 \lesssim A^2 \epsilon'^4,$$

where the indices in the above sums satisfy $(k_0, i_0), (k_1, i_1), (k_a, i_a) \in H$. Next, we use the fact that the constraint equation (2.19) implies $Ri\not{e}m = R + \psi\psi$, so we bound:

$$|u|^{2q} \int_{-v\underline{v}}^{u} \int_{-\frac{\hat{u}}{\underline{v}}}^{v} \int_{S^n} |\hat{v}| w_{ml}^2 \left| \sum_{k+2j=m-1} \nabla^k \left(Ri\not{e}m^{j+1} \nabla_3^l \underline{\Psi}^G \right) \right|^2 d\mathring{V}ol d\hat{v} d\hat{u} \lesssim$$

$$\lesssim |u|^{2q} \int_{-v\underline{v}}^{u} \int_{-\frac{\hat{u}}{\underline{v}}}^{v} \sum_{(k_0,l)\in H} \hat{v}^{2+2l-2p} |\hat{u}|^{2p-2q} \left\| \nabla_3^l \underline{\Psi}^G \right\|_{\widetilde{H}^{k_0}(S_{\hat{u},\hat{v}})}^2 \prod_{a=1}^{j+1} \left(\sum_{(k_a,0)\in L} \hat{v}^4 \left\| Ri\not{e}m \right\|_{\widetilde{H}^{k_a}(S_{\hat{u},\hat{v}})}^2 \right)$$

$$\lesssim \underline{v} \cdot \sum_{(k_0,l)\in H} \left\| \underline{\Psi}^G \right\|_{\mathcal{C}_{k_0,l}}^2 \lesssim \underline{v} \cdot A\epsilon'^2.$$

As before, a very simple modification of this argument allows us to also bound the corresponding term with $Ri\not{e}m$ in $Err_{ml}^{\underline{\nu}}$. This completes bounding the error terms for our first energy estimate. In particular, we improved the bootstrap assumption for $\left\| \underline{\alpha} \right\|_{\mathcal{C}_{m,l}}^2$ and the last two terms in $\left\| \underline{\nu} \right\|_{\mathcal{C}_{m,l}}^2$. We point out that we already have good control of the bulk term for $\underline{\nu}$, which will help us in the next energy estimate.

Step 2. The Bianchi pair $(R, \underline{\nu})$. Similarly to the derivation of equations (3.27)-(3.28), we use the signature considerations in Lemma 2.3 to rewrite the Bianchi equations (2.26)-(2.27) as:

$$\nabla_3 R_{ABCD} = -2\nabla_{[A}\underline{\nu}_{|CD|B]} + \psi\Psi,\ \nabla_4\underline{\nu}_{ABC} = -2\nabla_{[A}\tau_{B]C} + \psi\underline{\Psi}^G. \tag{3.33}$$

Step 2a. The system of commuted equations. For any $(m, l) \in H$, we commute the equations (3.33) with $\nabla^m\nabla_3^l$ using Lemmas 2.4, 2.6, and 2.7:

$$\begin{aligned}\nabla_3\nabla^m\nabla_3^l R_{ABCD} = &- 2\nabla_{[A}\nabla^m\nabla_3^l\underline{\nu}_{|CD|B]} + \underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\Psi) + \\ &+ \underline{\mathcal{F}}_{(m)(l)(m+l)}(\Psi) + \sum_{i+2j=m-1}\nabla^i\big(Ri\not{e}m^{j+1}\nabla_3^l\underline{\Psi}^G\big),\end{aligned}$$

$$\begin{aligned}\nabla_4\nabla^m\nabla_3^l\underline{\nu}_{ABC} = &- 2\nabla_{[A}\nabla^m\nabla_3^l\tau_{B]C} + \underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\underline{\Psi}^G) + \\ &+ \underline{\mathcal{F}}_{(m)(l)(m+l)}(\underline{\Psi}^G) + \sum_{i+2j=m-1}\nabla^i\big(Ri\not{e}m^{j+1}\nabla_3^l\underline{\Psi}^G\big).\end{aligned}$$

We conjugate the equations with the weight $w_{ml} = v^{\frac{3}{2}+m+l-p}|u|^{p-q}$:

$$\nabla_3 w_{ml}\nabla^m\nabla_3^l R_{ABCD} + \frac{p-q}{|u|}w_{ml}\nabla^m\nabla_3^l R_{ABCD} = -2\nabla_{[A}w_{ml}\nabla^m\nabla_3^l\underline{\nu}_{|CD|B]} + w_{ml}Err^R_{ml}, \tag{3.34}$$

$$\nabla_4 w_{ml}\nabla^m\nabla_3^l\underline{\nu}_{ABC} = -2\nabla_{[A}w_{ml}\nabla^m\nabla_3^l\tau_{B]C} + w_{ml}\underline{Err}^{\nu}_{ml} + O\bigg(\frac{w_{ml}}{v}\cdot\big|\nabla^m\nabla_3^l\underline{\nu}\big|\bigg), \tag{3.35}$$

where we have the error terms:

$$Err^R_{ml} = \underline{\mathcal{F}}_{(m)(l)(m+l)}(\Psi) + \underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\Psi) + \sum_{i+2j=m-1}\nabla^i\big(Ri\not{e}m^{j+1}\nabla_3^l\underline{\Psi}^G\big),$$

$$\underline{Err}^{\nu}_{ml} = \underline{\mathcal{F}}_{(m)(l)(m+l)}(\underline{\Psi}^G) + \underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\underline{\Psi}^G) + \sum_{i+2j=m-1}\nabla^i\big(Ri\not{e}m^{j+1}\nabla_3^l\underline{\Psi}^G\big).$$

Step 2b. The energy estimates for $(R, \underline{\nu})$. We denote $\mathcal{D} = w_{ml}\nabla^m\nabla_3^l$. We notice that integration by parts and the constraint equation (2.20) give:

$$\begin{aligned}&\int_{S^n}\Big(-2\nabla_{[A}\mathcal{D}\underline{\nu}_{|CD|B]}\cdot\mathcal{D}R^{ABCD} - 4\mathcal{D}\underline{\nu}^{ABC}\cdot\nabla_{[A}\mathcal{D}\tau_{B]C}\Big)d\mathring{V}ol = \\ &= \int_{S^n}2\mathcal{D}\underline{\nu}^{ABC}\cdot\Big(\nabla^D\mathcal{D}R_{DCAB} - 2\nabla_{[A}\mathcal{D}\tau_{B]C}\Big)d\mathring{V}ol + O\bigg(v^{-1}\int_{S^n}\big|\mathcal{D}R\big|\cdot\big|\mathcal{D}\underline{\nu}\big|d\mathring{V}ol\bigg) \\ &= O\bigg(\int_{S^n}\big|\mathcal{D}\underline{\nu}\big|\cdot\Big(v^{-1}\big|\mathcal{D}R\big| + w_{ml}\big|\underline{Err}^{\nu}_{ml}\big|\Big)d\mathring{V}ol\bigg).\end{aligned}$$

Proceeding as before, we obtain the energy estimate in region III for $v < \underline{v}$:

$$|u|^{2q}\int_{-v\underline{v}}^{u}\int_{S^n}w_{ml}^2\big|\nabla^m\nabla_3^l\underline{\nu}\big|^2 d\mathring{V}ol d\hat{u} + \int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n}w_{ml}^2\big|\nabla^m\nabla_3^l R\big|^2 d\mathring{V}ol d\hat{v} +$$

$$+|u|^{2q}\int_{-v\underline{v}}^{u}\int_{-\frac{\hat{u}}{\underline{v}}}^{v}\int_{S^n}\frac{w_{ml}^2}{|\hat{u}|}\left|\nabla^m\nabla_3^l R\right|^2\lesssim$$

$$\lesssim \underline{v}^{2p-2q}|u|^{2q}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n}\hat{v}^{3+2m+2l-2q}\Big(\left|\nabla^m\nabla_3^l R\right|^2+\left|\nabla^m\nabla_3^l\underline{\nu}\right|^2\Big)(-\underline{v}\hat{v},\hat{v})d\mathring{V}old\hat{v}+$$

$$+|u|^{2q}\int_{-v\underline{v}}^{u}\int_{-\frac{\hat{u}}{\underline{v}}}^{v}\int_{S^n}\left(|\hat{u}|w_{ml}^2\left|Err_{ml}^{R}\right|^2+w_{ml}^2\left|\nabla^m\nabla_3^l\underline{\nu}\right|\cdot\left|\underline{Err}_{ml}^{\nu}\right|+\frac{w_{ml}^2}{\hat{v}}\left|\nabla^m\nabla_3^l\underline{\nu}\right|^2\right)d\mathring{V}old\hat{v}d\hat{u}$$

$$\lesssim \underline{v}^{2p}\epsilon'^2+\underline{v}\cdot A\epsilon'^2+|u|^{2q}\int_{-v\underline{v}}^{u}\int_{-\frac{\hat{u}}{\underline{v}}}^{v}\int_{S^n}\Big(|\hat{u}|w_{ml}^2\left|Err_{ml}^{R}\right|^2+\hat{v}w_{ml}^2\left|\underline{Err}_{ml}^{\nu}\right|^2\Big)d\mathring{V}old\hat{v}d\hat{u},$$

where we used the fact that we already control the bulk term for $\underline{\nu}$ from the previous energy estimate, and we bounded the data term as before. Moreover, we remark that each term of $\hat{v}w_{ml}^2\left|\underline{Err}_{ml}^{\nu}\right|^2$ is contained in the error terms of $\hat{v}w_{ml}^2\left|Err_{ml}^{\alpha}\right|^2$, and similarly each term of $|\hat{u}|w_{ml}^2\left|Err_{ml}^{R}\right|^2$ is contained in the error terms of $|\hat{u}|w_{ml}^2\left|Err_{ml}^{\nu}\right|^2$. Therefore, the bounds on the error terms in the previous energy estimate also allow us to bound the right hand side of the above estimate by $\underline{v}\cdot A\epsilon'^2+\epsilon'^2$. This improves the bootstrap assumption for $\left\|\underline{\nu}\right\|_{\mathcal{C}_{m,l}}^2$ and the last two terms in $\left\|R\right\|_{\mathcal{C}_{m,l}}^2$.

The same argument applies in the case of the Bianchi pairs (R,ν) and (ν,α), where at each step we use the control of the bulk terms from the previous step and proceed as above. As a result, we can improve the bootstrap assumption on the high regularity curvature norm (3.15) and prove that:

$$\mathcal{C}_{\widetilde{u},\widetilde{v}}\lesssim\epsilon'^2+\underline{v}\cdot A\epsilon'^2,$$

which implies the desired estimate (3.26). $\square$

3.2.4 Estimates for Low Regularity Curvature Components

In this section, we show improved bounds on the low regularity curvature components (3.16) by proving:

Proposition 3.4. *Assuming that the bootstrap assumption* (3.19) *holds, there exists a constant* $C\ll A$, *such that we have the improved estimate for the low regularity curvature components* (3.16)*:*

$$\mathcal{L}_{\widetilde{u},\widetilde{v}}\leq C\epsilon'^2. \tag{3.36}$$

Proof. Step 1. Improving the bounds for $\underline{\alpha}$. For any $(m,l)\in L$, we consider the equation (3.30) satisfied by $\underline{\alpha}$ conjugated by $x_{ml}=v^{2+m+l-p}|u|^p$:

$$\nabla_4 x_{ml}\nabla^m\nabla_3^l\underline{\alpha}_{AB}+\left(\frac{n-4}{2}-l+p\right)\frac{x_{ml}}{v}\nabla^m\nabla_3^l\underline{\alpha}_{AB}= \tag{3.37}$$

$$=x_{ml}\nabla^{m+1}\nabla_3^l\underline{\nu}+x_{ml}\widetilde{Err}^{\underline{\alpha}}_{ml}+O\bigg(\sum_{i=0}^{l-1}\frac{x_{mi}}{v}\cdot\left|\nabla^m\nabla_3^i\underline{\alpha}\right|+\underline{v}\cdot\frac{x_{ml}}{v}\cdot\left|\nabla^m\nabla_3^l\underline{\alpha}\right|\bigg), \tag{3.38}$$

where the implicit constant in the last term is independent of p, and the error term is defined by:

$$\begin{aligned}\widetilde{Err}^{\underline{\alpha}}_{ml}=&\underline{\mathcal{F}}_{(m)(l)(m+l)}(\underline{\Psi}^G)+\underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\underline{\Psi}^G) \\ &+\underline{\mathcal{F}}'_{(m)(l)(m+l)}(\Psi)+\sum_{i+2j=m-1}\nabla^i\big(Ri\not{e}m^{j+1}\nabla_3^l\underline{\nu}\big).\end{aligned} \tag{3.39}$$

Step 1a. The energy estimates for $\underline{\alpha}$. We contract equation (3.37) by $x_{ml}\nabla^m\nabla_3^l\underline{\alpha}$, and integrate in v. We use the good bulk term obtained and Cauchy-Schwarz, together with a brief induction on l argument needed to bound the second to last term in (3.38). Summing for all $(m,l)\in L$, we obtain the estimate in region III for $v<\underline{v}$:

$$\sum_{(m,l)\in L}\int_{S^n}x_{ml}^2\left|\nabla^m\nabla_3^l\underline{\alpha}\right|^2(u,v)d\mathring{V}ol+\sum_{(m,l)\in L}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n}\frac{x_{ml}^2}{\hat{v}}\left|\nabla^m\nabla_3^l\underline{\alpha}\right|^2(u,\hat{v})d\mathring{V}old\hat{v}\lesssim$$

$$\lesssim\sum_{(m,l)\in L}\int_{S^n}x_{ml}^2\left|\nabla^m\nabla_3^l\underline{\alpha}\right|^2(u,-\underline{v}^{-1}u)+\sum_{(m,l)\in L}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n}\hat{v}x_{ml}^2\left|\nabla^{m+1}\nabla_3^l\underline{\nu}\right|^2$$

$$+\sum_{(m,l)\in L}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n}\hat{v}x_{ml}^2\left|\widetilde{Err}^{\underline{\alpha}}_{ml}\right|^2.$$

The data term is bounded using Proposition 3.2 by:

$$\sum_{(m,l)\in L}\int_{S^n}x_{ml}^2\left|\nabla^m\nabla_3^l\underline{\alpha}\right|^2(u,-\underline{v}^{-1}u)d\mathring{V}ol\lesssim\underline{v}^{2p}\epsilon'^2.$$

Since $vx_{ml}^2=|u|^{2q}w_{(m+1)l}^2$, we notice that the second term is bounded by $\left\|\underline{\nu}\right\|^2_{\mathcal{C}_{(m+1)l}}$, which is controlled by $\underline{v}\cdot A\epsilon'^2$ according to Proposition 3.3. We obtain the estimate:

$$\sum_{(m,l)\in L}\int_{S^n}x_{ml}^2\left|\nabla^m\nabla_3^l\underline{\alpha}\right|^2(u,v)d\mathring{V}ol+\sum_{(m,l)\in L}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n}\frac{x_{ml}^2}{\hat{v}}\left|\nabla^m\nabla_3^l\underline{\alpha}\right|^2(u,\hat{v})d\mathring{V}old\hat{v}\lesssim$$

$$\lesssim\epsilon'^2+\underline{v}\cdot A\epsilon'^2+\sum_{(m,l)\in L}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n}\left(\hat{v}x_{ml}^2\left|\underline{\mathcal{F}}_{(m)(l)(m+l)}(\underline{\Psi}^G)\right|^2+\hat{v}x_{ml}^2\left|\underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\underline{\Psi}^G)\right|^2\right)+$$

$$+\sum_{(m,l)\in L}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n}\hat{v}x_{ml}^2\left|\underline{\mathcal{F}}'_{(m)(l)(m+l)}(\Psi)\right|^2+\sum_{(m,l)\in L}\sum_{i+2j=m-1}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n}\hat{v}x_{ml}^2\left|\nabla^i\big(Ri\not{e}m^{j+1}\nabla_3^l\underline{\nu}\big)\right|^2.$$

Step 1b. Bounding the error terms. As in the proof of Proposition 3.3, we remark that:

$$\sum_{(m,l)\in L} x_{ml}^2 \big|\underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\underline{\Psi}^G)\big|^2 \lesssim \sum_{(m,l)\in L} x_{ml}^2 \big|\underline{\mathcal{F}}_{(m)(l)(m+l)}(\underline{\Psi}^G)\big|^2,$$

so bounding the first term in $\widetilde{Err}_{ml}^{\underline{\alpha}}$ will also imply control of the second term. We bound the first term:

$$|u|^{2q}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n} \hat{v}^2 w_{ml}^2 \big|\underline{\mathcal{F}}_{(m)(l)(m+l)}(\underline{\Psi}^G)\big|^2 \lesssim \sum_{\substack{i+j+k\leq m+l\\ i\leq l,k\leq m}} |u|^{2q}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n} \hat{v}^2 w_{ml}^2 \big|\nabla^k\nabla_3^i\big(\psi^{j+1}\underline{\Psi}^G\big)\big|^2$$

$$\lesssim |u|^{2q}\int_{-\frac{u}{\underline{v}}}^{v}\sum_{(k_0,i_0)\in L}\hat{v}^{3+2i_0-2p}|u|^{2p-2q}\Big\|\nabla_3^{i_0}\underline{\Psi}^G\Big\|^2_{\widetilde{H}^{k_0}(S_{u,\hat{v}})}\cdot\prod_{a=1}^{j+1}\sum_{(k_a,i_a)\in L}\hat{v}^{2+2i_a}\Big\|\nabla_3^{i_a}\psi\Big\|^2_{\widetilde{H}^{k_a}(S_{u,\hat{v}})}$$

$$\lesssim \sum_{(k,i)\in L}\big\|\underline{\Psi}^G\big\|^2_{\mathcal{C}_{k,i}} \lesssim \underline{v}\cdot A\epsilon'^2,$$

where we used the improved estimates from Proposition 3.3. Next, we have:

$$|u|^{2q}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n}\hat{v}^2 w_{ml}^2\big|\underline{\mathcal{F}}'_{(m)(l)(m+l)}(\Psi)\big|^2 d\mathring{V}ol d\hat{v} \lesssim$$

$$\lesssim \mathcal{R}_{u,v}\cdot\sum_{(k,i)\in L}\big\|\underline{\Psi}^G\big\|^2_{\mathcal{C}_{k,i}} + A\epsilon'^2\sum_{(k,i)\in L}|u|^{2q}\int_{-\frac{u}{\underline{v}}}^{v}\hat{v}^{3+2i-2p}|u|^{2p-2q}\Big\|\nabla_3^i\underline{\alpha}\Big\|^2_{\widetilde{H}^k(S_{u,\hat{v}})}d\hat{v}.$$

The second term can be absorbed on the left hand side of the estimate using the good bulk term. We bound the last error term using the fact that the constraint equation (2.19) implies $Ri\not{e}m = R + \psi\psi$:

$$\sum_{i+2j=m-1}|u|^{2q}\int_{-\frac{u}{\underline{v}}}^{v}\int_{S^n}\hat{v}^2 w_{ml}^2\Big|\nabla^i\big(Ri\not{e}m^{j+1}\nabla_3^l\underline{\nu}\big)\Big|^2 d\mathring{V}ol d\hat{v} \lesssim$$

$$\lesssim |u|^{2q}\int_{-\frac{\hat{u}}{\underline{v}}}^{v}\sum_{(k_0,l)\in L}\hat{v}^{3+2l-2p}|\hat{u}|^{2p-2q}\Big\|\nabla_3^l\underline{\Psi}^G\Big\|^2_{\widetilde{H}^{k_0}}\prod_{a=1}^{j+1}\Bigg(\sum_{(k_a,0)\in L}\hat{v}^4\Big\|Ri\not{e}m\Big\|^2_{\widetilde{H}^{k_a}}\Bigg)$$

$$\lesssim \sum_{(k,l)\in L}\big\|\underline{\Psi}^G\big\|^2_{\mathcal{C}_{k,l}} \lesssim \underline{v}A\epsilon'^2.$$

Thus, we improved the bootstrap assumption for $\big\|\underline{\alpha}\big\|^2_{\mathcal{L}_{m,l}}$ for any $(m,l)\in L$.

Step 2. Improving the bounds for $\underline{\Psi}^G$*.* Using the Bianchi equations in Proposition 2.6, the signature considerations in Lemma 2.3, and the commutation Lemmas 2.4-2.7, we obtain that the curvature components $\underline{\Psi}^G$ satisfy the schematic equation:

$$\nabla_3 v^{2+m+l}\nabla^m\nabla_3^l\underline{\Psi}^G = v^{2+m+l}Err_{ml}, \tag{3.40}$$

where we have the error term:

$$Err_{ml} = \nabla^{m+1}\nabla_3^l \Psi^G + \underline{\mathcal{F}}_{(m)(l)(m+l)}(\Psi) + \underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\Psi).$$

We use [RSR23, Lemma 9.6] and the bounds in Proposition 3.2 to obtain the estimate:

$$\sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}} \int_{S^n} v^{4+2m+2l} \left|\nabla^m \nabla_3^l \underline{\Psi}^G\right|^2 d\mathring{V}ol \lesssim$$

$$\lesssim \epsilon'^2 + \sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}} \left(\int_{-v\underline{v}}^{u} v^{2+m+l} \left(\int_{S^n} \left|Err_{ml}\right|^2 d\mathring{V}ol \right)^{\frac{1}{2}} d\hat{u} \right)^2$$

$$\lesssim \epsilon'^2 + \sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}} \underline{v}^{1-100p} \sup_{\mathring{u}\in[-v\underline{v},u]} |\mathring{u}|^{100q} \int_{-v\underline{v}}^{\mathring{u}} \int_{S^n} v^{5+2m+2l-100p} |\hat{u}|^{100p-100q} \left|Err_{ml}\right|^2 d\mathring{V}ol d\hat{u}.$$

Since $\Psi^G \neq \alpha$, the first error term is bounded by:

$$\sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}} \underline{v}^{1-100p} \sup_{\mathring{u}\in[-v\underline{v},u]} |\mathring{u}|^{100q} \int_{-v\underline{v}}^{\mathring{u}} \int_{S^n} v^{5+2m+2l-100p} |\hat{u}|^{100p-100q} \left|\nabla^{m+1}\nabla_3^l \Psi^G\right|^2 \lesssim \underline{v}^{1-100p} \mathcal{C}_{\widetilde{u},\widetilde{v}}.$$

Arguing as before, bounding the second error term will also imply bounds for the third error term once summing. We have the bound:

$$\underline{v}^{1-100p} \sup_{\mathring{u}\in[-v\underline{v},u]} |\mathring{u}|^{100q} \int_{-v\underline{v}}^{\mathring{u}} \int_{S^n} v^{5+2m+2l-100p} |\hat{u}|^{100p-100q} \left|\underline{\mathcal{F}}_{(m)(l)(m+l)}(\Psi)\right|^2 d\mathring{V}ol d\hat{u}$$

$$\lesssim \mathcal{L}_{u,v} \cdot \underline{v}^{1-100p} \sup_{\mathring{u}\in[-v\underline{v},u]} |\mathring{u}|^{100q} \int_{-v\underline{v}}^{\mathring{u}} v^{-1-98p} |\hat{u}|^{98p-100q} d\hat{u} \lesssim \underline{v} \cdot \mathcal{L}_{u,v} \lesssim \underline{v} \cdot A\epsilon'^2.$$

As a result, we improve the bootstrap assumption on the low regularity curvature norm (3.16) and prove that:

$$\mathcal{L}_{\widetilde{u},\widetilde{v}} \lesssim \epsilon'^2 + \underline{v} \cdot A\epsilon'^2,$$

which implies the desired estimate (3.36). □

3.2.5 Estimates for Ricci Coefficients

In this section, we show improved bounds on the Ricci coefficients (3.17) by proving:

Proposition 3.5. *Assuming that the bootstrap assumption* (3.19) *holds, there exists a constant $C \ll A$, such that we have the improved estimate for the Ricci coefficients* (3.17)*:*

$$\mathcal{R}_{\widetilde{u},\widetilde{v}} \leq C\epsilon'^2. \tag{3.41}$$

Proof. We notice that according to Proposition 2.4, the Ricci coefficients satisfy schematic equations:

$$\nabla_3\big(\hat{\chi}, \underline{\hat{\chi}}\big) = \Psi^G + \psi\psi^*, \ \nabla_3\big(\mathrm{tr}\chi^*, \mathrm{tr}\underline{\chi}^*\big) = \psi\psi^*. \tag{3.42}$$

We allow the curvature term on the right hand side in order to treat all the equations at the same time. For any $(m, l) \in H$, we commute the equations (3.42) with $\nabla^m\nabla_3^l$:

$$\nabla_3 v^{1+m+l}\nabla^m\nabla_3^l\psi^* = v^{1+m+l}Err_{ml}, \tag{3.43}$$

where we have the error term:

$$Err_{ml} = \nabla^m\nabla_3^l\Psi^G + \underline{\mathcal{F}}_{(m)(l)(m+l)}(\psi^*),$$

and we define $\underline{\mathcal{F}}_{(m)(l)(m+l)}(\psi^*)$ just as $\underline{\mathcal{F}}_{(m)(l)(m+l)}(\Psi)$ in Definition 2.4, but replacing Ψ with ψ^*. We integrate (3.43) and apply [RSR23, Lemma 9.6] as in the proof of Proposition 3.4 to obtain:

$$\sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}} \int_{S^n} v^{2+2m+2l}\big|\nabla^m\nabla_3^l\psi^*\big|^2 d\mathring{V}ol \lesssim$$

$$\epsilon'^2 + \sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}} \left(\int_{-v\underline{v}}^{u} v^{1+m+l}\left(\int_{S^n}\big|Err_{ml}\big|^2 d\mathring{V}ol\right)^{\frac{1}{2}} d\hat{u}\right)^2$$

$$\lesssim \epsilon'^2 + \sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}} \underline{v}^{1-100p} \sup_{\mathring{u}\in[-v\underline{v},u]} |\mathring{u}|^{100q} \int_{-v\underline{v}}^{\mathring{u}}\int_{S^n} v^{3+2m+2l-100p}|\hat{u}|^{100p-100q}\big|Err_{ml}\big|^2 d\mathring{V}ol d\hat{u}.$$

Since $\Psi^G \neq \alpha$, the first error term is bounded by:

$$\sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}} \underline{v}^{1-100p} \sup_{\mathring{u}\in[-v\underline{v},u]} |\mathring{u}|^{100q} \int_{-v\underline{v}}^{\mathring{u}}\int_{S^n} v^{3+2m+2l-100p}|\hat{u}|^{100p-100q}\big|\nabla^m\nabla_3^l\Psi^G\big|^2 \lesssim \underline{v}^{1-100p}\mathcal{C}_{\widetilde{u},\widetilde{v}}.$$

We have the bound for the second error term:

$$\underline{v}^{1-100p} \sup_{\mathring{u}\in[-v\underline{v},u]} |\mathring{u}|^{100q} \int_{-v\underline{v}}^{\mathring{u}}\int_{S^n} v^{3+2m+2l-100p}|\hat{u}|^{100p-100q}\big|\underline{\mathcal{F}}_{(m)(l)(m+l)}(\psi^*)\big|^2 d\mathring{V}ol d\hat{u}$$

$$\lesssim \mathcal{R}_{u,v}\cdot\underline{v}^{1-100p} \sup_{\mathring{u}\in[-v\underline{v},u]} |\mathring{u}|^{100q}\int_{-v\underline{v}}^{\mathring{u}} v^{-1-98p}|\hat{u}|^{98p-100q}d\hat{u} \lesssim \underline{v}\cdot A\epsilon'^2.$$

As a result, we improve the bootstrap assumption on the Ricci coefficients norm (3.17) and prove that:

$$\mathcal{R}_{\widetilde{u},\widetilde{v}} \lesssim \epsilon'^2 + \underline{v}\cdot A\epsilon'^2,$$

which implies (3.41). □

Remark 3.10. *We can also prove an estimate for* $\nabla_3^{\frac{n-2}{2}}\psi^*$, *even though this term was not needed in the bootstrap argument. For any* $i \leq N_3 - 1$, *we have that* $\left(i, \frac{n-4}{2}\right) \in L$, *and the equation:*

$$\nabla^i \nabla_3^{\frac{n-2}{2}} \psi^* = \nabla^i \nabla_3^{\frac{n-4}{2}} \Psi + \nabla^i \nabla_3^{\frac{n-4}{2}} \big(\psi \psi^*\big). \tag{3.44}$$

Using the estimates in Proposition 3.4 and Proposition 3.5, we have that for all $i \leq N_3 - 1$:

$$\big\|\nabla^i \nabla_3^{\frac{n-2}{2}} \psi^*\big\|_{L^2(S_{u,v})} \lesssim \epsilon' |u|^{-p} \cdot |v|^{-1-i-\frac{n-2}{2}+p}. \tag{3.45}$$

3.2.6 Estimates for Metric Coefficients

In this section, we show improved bounds on the metric coefficients (3.18) by proving:

Proposition 3.6. *Assuming that the bootstrap assumption* (3.19) *holds, there exists a constant* $C \ll A$, *such that we have the improved estimate for the metric coefficients* (3.18)*:*

$$\mathcal{M}_{\widetilde{u},\widetilde{v}} \leq C \epsilon'^2. \tag{3.46}$$

Proof. The metric equations (2.7)-(2.8) imply that $\mathcal{L}_3 \mathcal{L}_\theta^m \not{g}^* = \mathcal{L}_\theta^m \psi^*$. We denote by Γ the Christoffel symbols of the metric $\not{g}$. We notice that the bootstrap assumption implies that for any $(m, 0) \in H$:

$$v^{2m} \big\|\mathcal{L}_\theta^{m-1} \Gamma\big\|_{L^2(S_{u,v})}^2 \lesssim 1. \tag{3.47}$$

For any horizontal tensor ϕ we have the formula $\mathcal{L}_\theta \phi = \nabla \phi + \Gamma \cdot \phi$, which implies by induction that:

$$\mathcal{L}_\theta^m \phi = \nabla^m \phi + \sum_{i+j+k=m-1} \mathcal{L}_\theta^i \big(\Gamma^{j+1}\big) \nabla^k \phi. \tag{3.48}$$

Thus, we obtain the equation:

$$\mathcal{L}_3 \mathcal{L}_\theta^m \not{g}^* = \nabla^m \psi^* + \sum_{i+j+k=m-1} \mathcal{L}_\theta^i \big(\Gamma^{j+1}\big) \nabla^k \psi^* =: Err_m.$$

We have the estimate for any $(m, 0) \in H$:

$$\sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}} v^{2m} \big\|\mathcal{L}_\theta^m \not{g}^*\big\|_{L^2(S_{u,v})}^2 \lesssim \epsilon'^2 + \sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}} \left(\int_{-v\underline{v}}^{u} v^m \left(\int_{S^n} \big|Err_m\big|^2 d\mathring{V}ol \right)^{\frac{1}{2}} d\hat{u} \right)^2$$

$$\lesssim \epsilon'^2 + \sup_{(u,v)\in P_{\widetilde{u},\widetilde{v}}} \underline{v}^{1-100p} \sup_{\mathring{u}\in[-v\underline{v},u]} |\mathring{u}|^{100q} \int_{-v\underline{v}}^{\mathring{u}} \int_{S^n} v^{1+2m-100p} |\hat{u}|^{100p-100q} \big|Err_m\big|^2 d\mathring{V}ol d\hat{u}.$$

The first error term can be bounded as usual by $\underline{v} \cdot A\epsilon'^2$. For the second term we have:

$$\sum_{i+j+k=m-1} \underline{v}^{1-100p} \sup_{\mathring{u}\in[-v\underline{v},u]} |\mathring{u}|^{100q} \int_{-v\underline{v}}^{\mathring{u}} \int_{S^n} v^{1+2k-100p} |\hat{u}|^{100p-100q} \big|\nabla^k \psi^*\big|^2 v^{2+2i+2j} \big|\mathcal{L}_\theta^i\big(\Gamma^{j+1}\big)\big|^2$$

$$\lesssim \underline{v}^{1-100p} \sup_{\mathring{u}\in[-v\underline{v},u]} |\mathring{u}|^{100q} \int_{-v\underline{v}}^{\mathring{u}} \sum_{(k,0)\in L} v^{1-100p} |\hat{u}|^{100p-100q} \|\psi^*\|^2_{\widetilde{H}^k} \prod_{a=1}^{j+1} \sum_{(k_a,0)\in L} \int_{S^n} v^{2+2k_a} \big|\mathcal{L}_\theta^{k_a}\Gamma\big|^2,$$

which is also bounded by $\underline{v} \cdot A\epsilon'^2$, allowing us to improve the bootstrap assumption on the metric coefficients norm (3.18) and prove that:

$$\mathcal{M}_{\widetilde{u},\widetilde{v}} \lesssim \epsilon'^2 + \underline{v} \cdot A\epsilon'^2,$$

which implies (3.46). ☐

3.2.7 Quantitative Bounds in Region III

In this section, we complete the bootstrap argument and we obtain the main result in region III on the existence of the solution satisfying quantitative bounds. We then complete the proof of Theorem 3.1 and we prove propagation of regularity in Theorem 3.2.

We first show that the bootstrap assumption (3.19) can be improved:

Proposition 3.7. *Let* $(\mathcal{M}, g)$ *be a spacetime obtained in Proposition 3.2, which exists in the characteristic triangle* $P_{\widetilde{u},\widetilde{v}}$ *contained in region III with* $\widetilde{v} \leq \underline{v}$*. We assume that the spacetime satisfies the bootstrap assumption* (3.19)*. There exists a constant* $C \ll A$ *such that the spacetime satisfies the improved bound:*

$$\mathcal{C}_{\widetilde{u},\widetilde{v}} + \mathcal{L}_{\widetilde{u},\widetilde{v}} + \mathcal{R}_{\widetilde{u},\widetilde{v}} + \mathcal{M}_{\widetilde{u},\widetilde{v}} \leq C\epsilon'^2. \tag{3.49}$$

Proof. Under the bootstrap assumption (3.19), we proved the improved estimates (3.26), (3.36), (3.41), and (3.46) in Propositions 3.3-3.6. We combine these estimates to obtain (3.49). ☐

We are now in the position to prove the main result in region III, establishing the existence of the solution satisfying quantitative bounds:

Proposition 3.8. *The solution of Proposition 3.2 can be extended uniquely as a regular straight self-similar vacuum solution in region III. Setting* $N_3 = N - 3\lfloor c_0/4 \rfloor n$*, and taking* $p > 0$ *to be a small constant, we have the bounds in region III with* $\{v \leq \underline{v}\}$*, for all* $0 \leq i \leq N_3$*, and all*

$0 \leq j \leq \frac{n-4}{2}$:

$$\left\|\nabla^i \nabla_3^j \underline{\alpha}\right\|_{L^2(S_{u,v})} \lesssim \epsilon^{1-2\delta} |u|^{-p} \cdot |v|^{-2-i-j+p}$$

$$\left\|\nabla^i \nabla_3^j \underline{\Psi}^G\right\|_{L^2(S_{u,v})} \lesssim \epsilon^{1-2\delta} |v|^{-2-i-j}$$

$$\left\|\nabla^i \nabla_3^j \psi^*\right\|_{L^2(S_{u,v})} \lesssim \epsilon^{1-2\delta} |v|^{-1-i-j}$$

$$\left\|\nabla^i \nabla_3^{\frac{n-2}{2}} \psi^*\right\|_{L^2(S_{u,v})} \lesssim \epsilon^{1-2\delta} |u|^{-p} \cdot |v|^{-1-i-\frac{n-2}{2}+p}, \ \textit{for } i \leq N_3 - 1$$

$$\left\|\mathcal{L}_\theta^i \slashed{g}^*\right\|_{L^2(S_{u,v})} \lesssim \epsilon^{1-2\delta} |v|^{-i}.$$

In addition, we have control of more detailed norms as proved in Propositions 3.3,3.4,3.5, and 3.6.

Proof. We recall that on the hypersurface $\{(u,v): \ u = -\underline{v}v, \ 0 < v \leq \underline{v}\}$ the solution satisfies (3.20). Using standard local existence results and the fact that according to Proposition 3.7 we can improve the bootstrap assumption (3.19), we obtain global existence in the region III. Moreover, the solution satisfies the bound (3.49) for all $(\widetilde{u}, \widetilde{v})$ in the region III with $\widetilde{v} \leq \underline{v}$. ☐

Proposition 3.8, together with Propositions 3.1 and 3.2, completes the proof of Theorem 3.1. To conclude this section, we prove the following propagation of regularity result:

Theorem 3.2. *Consider a global straight self-similar vacuum spacetime $(\mathcal{M}, g)$ which satisfies the hypothesis of Theorem 3.1. If the initial data $(\slashed{g}_0, h)$ is smooth, the spacetime $(\mathcal{M}, g)$ is also smooth.*

Proof. We explain how a slight modification of our previous arguments implies the proof of this result. We first consider smooth initial data $(\slashed{g}_0, h)$ satisfying (3.1), and prove self-similar estimates on the double null quantities for any angular regularity. For any $K \geq N_3, \ 0 < q \ll p \ll 1, \ 0 < \underline{v}_K \ll 1$, we denote by $\widetilde{\mathcal{C}}_K, \ \widetilde{\mathcal{L}}_K, \ \widetilde{\mathcal{R}}_K, \ \widetilde{\mathcal{M}}_K$ the norms defined in Section 3.2.1 with $(N_3, q, p, \underline{v})$ replaced by $(K, q, p, \underline{v}_K)$, and without the $*$ in the definitions of $\widetilde{\mathcal{R}}_K$ and $\widetilde{\mathcal{M}}_K$. We prove by induction that for any $K \geq N_3$, there exist $0 < \underline{v}_K \ll 1$ small enough and $C(K) > 0$ large enough such that for any $(\tilde{u}, \tilde{v}) \in \{0 \leq v \leq \underline{v}_K, \ -v\underline{v}_K \leq u \leq 0\}$:

$$\widetilde{\mathcal{C}}_K(\tilde{u}, \tilde{v}) + \widetilde{\mathcal{L}}_K(\tilde{u}, \tilde{v}) + \widetilde{\mathcal{R}}_K(\tilde{u}, \tilde{v}) + \widetilde{\mathcal{M}}_K(\tilde{u}, \tilde{v}) \leq C(K). \tag{3.50}$$

The base case $K = N_3$ was proved in Proposition 3.8. We assume that the induction hypothesis holds up to $K-1$ and prove it for K. The strategy is to split the spacetime into regions $I_K,\ II_K,\ III_K$, defined analogously to $I,\ II,\ III$ but with $\underline{v}$ replaced by $\underline{v}_K$.

In the region $I_K = \big\{ -1 \leq u \leq 0,\ 0 \leq v \leq -u\underline{v}_K \big\}$ we have propagation of regularity for the local existence result of [RSR18, Theorem 1.1]. For $0 < \underline{v}_K \ll 1$ small enough, we apply the regular estimates of [RSR18, Proposition 7.3] and we obtain that the estimates of Proposition 3.1 hold up to $K + 4c_0 n$ angular regularity, with the ϵ on the right hand side replaced by a constant $C_K^I > 0$ that is not assumed to be small.

In the region $II_K = \big\{ -1 \leq u \leq 0,\ -u\underline{v}_K \leq v \leq -u/\underline{v}_K \big\}$, the propagation of regularity follows by the argument of [RSR23, Section 7]. The idea is to conjugate the equations by $\exp(D_K \cdot v/u)$, for some large constant D_K. This argument is equivalent to using the Gronwall lemma. We obtain that the estimates of Proposition 3.2 hold up to $K + 3c_0 n$ angular regularity, once again with the ϵ on the right hand side replaced by a constant $C_K^{II} > 0$ that is not assumed to be small.

As a result, there exists a constant $C'_K > 0$ such that on $\big\{ v = -u/\underline{v}_K \big\}$ we have:

$$\widetilde{\mathcal{C}}_K(u, -u/\underline{v}_K) + \widetilde{\mathcal{L}}_K(u, -u/\underline{v}_K) + \widetilde{\mathcal{R}}_K(u, -u/\underline{v}_K) + \widetilde{\mathcal{M}}_K(u, -u/\underline{v}_K) \leq C'_K.$$

Moreover, we can assume that $C'_K \gg C(K-1)$. For some $A > 0$ large, depending on K, we make the bootstrap assumption in the region $III_K = \big\{ -1 \leq u \leq 0,\ -u/\underline{v}_K \leq v \leq \underline{v}_K \big\}$:

$$\widetilde{\mathcal{C}}_K(\tilde{u}, \tilde{v}) + \widetilde{\mathcal{L}}_K(\tilde{u}, \tilde{v}) + \widetilde{\mathcal{R}}_K(\tilde{u}, \tilde{v}) + \widetilde{\mathcal{M}}_K(\tilde{u}, \tilde{v}) \leq 2AC'_K. \tag{3.51}$$

We briefly explain how to improve the bootstrap assumption (3.51) in order to prove that:

$$\widetilde{\mathcal{C}}_K(\tilde{u}, \tilde{v}) + \widetilde{\mathcal{L}}_K(\tilde{u}, \tilde{v}) + \widetilde{\mathcal{R}}_K(\tilde{u}, \tilde{v}) + \widetilde{\mathcal{M}}_K(\tilde{u}, \tilde{v}) \leq AC'_K.$$

We repeat the proof of Proposition 3.7, with angular regularity given by K. Since all the quantities with angular regularity up to $K-1$ are already bounded by the induction hypothesis (3.50), the equations are linear in the top order terms that need to be bounded. Thus, each error term is a product of factors where at most one such factor is bounded using the bootstrap assumption (3.51), while the remaining factors are bounded using the induction hypothesis (3.50). In particular, at each step in the proof of Proposition 3.7 where we bounded an error term by $\lesssim \underline{v}^{\frac{1}{2}-100p} \cdot A \cdot \epsilon'^2$, we would now have the bound $\leq C\big(C(K-1)\big) + C\big(C(K-1)\big)\underline{v}_K^{\frac{1}{2}-100p} \cdot A \cdot C'_K$. We also notice that we had $O(n)$ top order error terms bounded by $\lesssim A^2 \cdot \epsilon'^4$, which would now be bounded by

$\leq C\big(C(K-1)\big) + \epsilon'^2 \cdot A \cdot C'_K \cdot O(n)$. Therefore, for $0 < \underline{v}_K \ll 1$ small enough we can improve the bootstrap assumption (3.51) as desired. This establishes the induction hypothesis (3.50) for K, with $C(K) = AC'_K$. Since the above bounds hold for all $K \geq N_3$, we obtain that the spacetime $\big(\mathcal{M}, g\big)$ is smooth. $\square$

4 Asymptotic Completeness

In this section we prove the second statement of Theorem 1.3, establishing asymptotic completeness. Given suitably small Cauchy initial data on a spacelike hypersurface in the sense of Remark 3.8, Theorem 3.1 implies global existence in the region $\{u < 0,\ v > 0\}$. We prove in Theorem 4.1 the existence of induced smooth scattering data at $\{u = 0\}$ and $\{v = 0\}$, which completes the proof of the second statement of Theorem 1.3. In the original $(n+1)$-dimensional formulation, this represents the proof of the second statement of Theorem 1.1. This section is based on [Cic24, Section 4].

Theorem 4.1. *Consider a global straight self-similar vacuum spacetime* $\big(\mathcal{M}, g\big)$ *which satisfies the conclusion of Theorem 3.1. There exists induced straight data* $\big(\underline{\not{g}_0}, \underline{h}\big)$ *at* $(u, v) = (0, 1)$*, such that* $\big(\mathcal{M}, g\big)$ *is the unique straight self-similar vacuum spacetime determined by this initial data, and moreover we have the estimates for* $0 \leq i \leq N'$*:*

$$\big\|\mathcal{L}_\theta^i\big(\underline{\not{g}_0} - \not{g}_{S^n}\big)\big\|_{L^2(S^n)} \lesssim \epsilon', \tag{4.1}$$

$$\big\|\nabla^i \underline{\mathcal{O}}\big\|_{L^2(S^n)} \lesssim \epsilon', \tag{4.2}$$

$$\big\|\nabla^i \underline{h}\big\|_{L^2(S^n)} \lesssim \epsilon', \tag{4.3}$$

where $N' = N - c_0 n$ *as in Theorem 3.1,* $\epsilon' = \epsilon^{1-2\delta}$*, and* $\underline{\mathcal{O}}$ *is the obstruction tensor of the metric* $\underline{\not{g}_0}$*.*

Moreover, if the spacetime $\big(\mathcal{M}, g\big)$ *is smooth the induced data* $\big(\underline{\not{g}_0}, \underline{h}\big)$ *is also smooth.*

We follow the strategy outlined in Section 1.3.2 of the introduction. We use the quantitative estimates on the global solution obtained in Theorem 3.1 in order to recover the induced scattering data at $\{u = 0\}$ and $\{v = 0\}$. By time orientation reversal symmetry, it suffices to prove the existence of induced asymptotic data at $\{u = 0\}$. We complete the proof Theorem 4.1 at the end of this section, using Propositions 4.1 and 4.2 below.

We first prove that certain regular quantities can be extended to $\{u=0\}$. These correspond to determining the first $\frac{n-2}{2}$ terms in the expansion of $\slashed{g}$ at $\{u=0\}$.

Proposition 4.1. *We have the following continuous extensions to* $\{u=0\}:$

1. *For any* $0 \leq l \leq \frac{n-4}{2}$ *and* $0 \leq m \leq N' + \frac{n-6}{2} - l$, *we have:*

$$\nabla^m \nabla_3^l \underline{\Psi}^G \in W_u^{1,1}\big([-1,0]\big) L^2(S^n).$$

Moreover, for any $\lambda > 0$ *we have the self-similarity relations on* $\{u=0\}:$

$$\nabla^m \nabla_3^l \big(\alpha_{AB}, \tau_{AB}\big)(0, \lambda v) = \lambda^{-l} \nabla^m \nabla_3^l \big(\alpha_{AB}, \tau_{AB}\big)(0, v),$$

$$\nabla^m \nabla_3^l \big(\nu_{ABC}, \underline{\nu}_{ABC}\big)(0, \lambda v) = \lambda^{1-l} \nabla^m \nabla_3^l \big(\nu_{ABC}, \underline{\nu}_{ABC}\big)(0, v),$$

$$\nabla^m \nabla_3^l R_{ABCD}(0, \lambda v) = \lambda^{2-l} \nabla^m \nabla_3^l R_{ABCD}(0, v),$$

and the self-similar bounds:

$$\big\| v^{2+m+l} \nabla^m \nabla_3^l \underline{\Psi}^G \big\|_{L^2(S^n)}\big|_{u=0} \lesssim \epsilon'. \tag{4.4}$$

2. *For any* $0 \leq l \leq \frac{n-6}{2}$ *and* $0 \leq m \leq N' + \frac{n-8}{2} - l$, *we have:*

$$\nabla^m \nabla_3^l \underline{\alpha} \in W_u^{1,1}\big([-1,0]\big) L^2(S^n).$$

Moreover, for any $\lambda > 0$ *we have the self-similarity relation on* $\{u=0\}:$

$$\nabla^m \nabla_3^l \underline{\alpha}_{AB}(0, \lambda v) = \lambda^{-l} \nabla^m \nabla_3^l \underline{\alpha}_{AB}(0, v),$$

and the self-similar bounds:

$$\big\| v^{2+m+l} \nabla^m \nabla_3^l \underline{\alpha} \big\|_{L^2(S^n)}\big|_{u=0} \lesssim \epsilon', \tag{4.5}$$

$$v^{2+m} \Big\| v^l \nabla^m \nabla_3^l \underline{\alpha} - \big(v^l \nabla^m \nabla_3^l \underline{\alpha}\big)\big|_{u=0} \Big\|_{L^2(S^n)} \lesssim \epsilon' \cdot \left|\frac{u}{v}\right|^{1-p}. \tag{4.6}$$

3. *For any* $0 \leq l \leq \frac{n-4}{2}$ *and* $0 \leq m \leq N' + \frac{n-4}{2} - l$, *we have:*

$$\nabla^m \nabla_3^l \psi \in W_u^{1,1}\big([-1,0]\big) L^2(S^n).$$

Moreover, for any $\lambda > 0$ *we have the self-similarity relations on* $\{u=0\}:$

$$\nabla^m \nabla_3^l \big(\chi_{AB}, \underline{\chi}_{AB}\big)(0, \lambda v) = \lambda^{1-l} \nabla^m \nabla_3^l \big(\chi_{AB}, \underline{\chi}_{AB}\big)(0, v),$$

and the self-similar bound:

$$\left\|v^{1+m+l}\nabla^m\nabla_3^l\psi^*\right\|_{L^2(S^n)}\big|_{u=0}\lesssim\epsilon'. \tag{4.7}$$

4. *For any* $0\leq l\leq\frac{n-2}{2}$ *and* $0\leq m\leq N'+\frac{n-4}{2}-l$*, we have:*

$$v^{m+l}\mathcal{L}_\theta^m\mathcal{L}_3^l\slashed{g}\in W_u^{1,1}\big([-1,0]\big)L^2(S^n).$$

Moreover, for any $\lambda>0$ *we have the self-similarity relation on* $\{u=0\}$:

$$\mathcal{L}_\theta^m\mathcal{L}_3^l\slashed{g}_{AB}(0,\lambda v)=\lambda^{2-l}\mathcal{L}_\theta^m\mathcal{L}_3^l\slashed{g}_{AB}(0,v),$$

and the self-similar bound:

$$\left\|v^{m+l}\mathcal{L}_\theta^m\mathcal{L}_3^l\slashed{g}^*\right\|_{L^2(S^n)}\big|_{u=0}\lesssim\epsilon'. \tag{4.8}$$

Proof. We first restrict to the region $\{v\leq\underline{v}\}$. Thus, we can use the estimates in Proposition 3.8. We recall that in the proof of Proposition 3.4 we had for any $(m,l)\in L$:

$$\nabla_3v^{2+m+l}\nabla^m\nabla_3^l\underline{\Psi}^G=v^{2+m+l}Err_{ml},$$

where the error term satisfies:

$$\left\|v^{2+m+l}Err_{ml}\right\|_{L_u^1([-v\underline{v},0))L^2(S^n)}\lesssim\epsilon'.$$

We also have that $\partial_u\big(v^{2+m+l}\nabla^m\nabla_3^l\underline{\Psi}^G\big)=\nabla_3v^{2+m+l}\nabla^m\nabla_3^l\underline{\Psi}^G+v^{2+m+l}\underline{\chi}\cdot\nabla^m\nabla_3^l\underline{\Psi}^G$. From the proof of Proposition 3.5, we have that:

$$\left\|v^{2+m+l}\underline{\chi}\cdot\nabla^m\nabla_3^l\underline{\Psi}^G\right\|_{L_u^1([-v\underline{v},0))L^2(S^n)}\lesssim\epsilon'.$$

As a result, we obtain that $v^{2+m+l}\nabla^m\nabla_3^l\underline{\Psi}^G\in W_u^{1,1}\big([-v\underline{v},0]\big)L^2(S^n)$ and that (4.4) holds. Using self-similarity, we can extend these results from the region $\{v\leq\underline{v}\}$ to all $v>0$. Moreover, by continuity we obtain that the self-similarity relations also hold along $u=0$.

In order to prove the second statement, we notice that for any $0\leq l\leq\frac{n-6}{2}$ and $0\leq m\leq N'+\frac{n-8}{2}-l$:

$$\partial_u\big(v^{2+m+l}\nabla^m\nabla_3^l\underline{\alpha}\big)=v\psi\cdot v^{1+m+l}\nabla^m\nabla_3^l\underline{\alpha}+v^{2+m+l}\nabla^m\nabla_3^{l+1}\underline{\alpha}+\sum_{i=0}^m v^{1+i}\nabla^i\psi\cdot v^{1+m-i+l}\nabla^{m-i}\nabla_3^l\underline{\alpha}.$$

Since $(m,l+1)\in L$, we can bound the above terms using our previous estimates and we obtain:

$$\left\|\partial_u\big(v^{2+m+l}\nabla^m\nabla_3^l\underline{\alpha}\big)\right\|_{L^2(S^n)}\lesssim\epsilon'\cdot|u|^{-p}v^{-1+p}. \tag{4.9}$$

This implies that $v^{2+m+l}\nabla^m\nabla_3^l\underline{\alpha} \in W_u^{1,1}\big([-v\underline{v},0]\big)L^2(S^n)$, that (4.5)-(4.6) hold. As before, the self-similarity relation extends to $u=0$ by continuity.

The proof of the third statement follows exactly as the first statement, but using the corresponding estimates from the proof of Proposition 3.5.

We prove the fourth statement in the case of $\mathcal{L}_\theta^m\mathcal{L}_3^{l+1}\not{g}$, since the case of no $\mathcal{L}_3$ derivatives is similar. We compute the commuted equation for any $(m,l)\in H$ using (3.48) and its analogue in the e_3 direction:

$$\mathcal{L}_3\big(\mathcal{L}_\theta^m\mathcal{L}_3^{l+1}\not{g}^*\big) = 2\mathcal{L}_\theta^m\mathcal{L}_3^{l+1}\underline{\chi}^* = \mathcal{L}_\theta^m\mathcal{L}_3^l\big(\underline{\alpha}+\psi\psi^*\big) = \mathcal{L}_\theta^m\big(\nabla_3^l\underline{\alpha}+\underline{\mathcal{F}}_{(0)(l-1)(l-1)}(\underline{\alpha})+\underline{\mathcal{F}}_{(0)(l)(l)}(\psi^*)\big)$$

$$= \nabla^m\nabla_3^l\underline{\alpha} + \sum_{i+j+k=m-1}\mathcal{L}_\theta^i\big(\Gamma^{j+1}\big)\nabla^k\nabla_3^l\underline{\alpha} + \sum_{i+j+k=m-1}\mathcal{L}_\theta^i\big(\Gamma^{j+1}\big)\underline{\mathcal{F}}_{(k)(l-1)(k+l-1)}(\underline{\alpha})+$$

$$+\underline{\mathcal{F}}_{(m)(l-1)(m+l-1)}(\underline{\alpha})+\underline{\mathcal{F}}_{(m)(l)(m+l)}(\psi^*)+\sum_{i+j+k=m-1}\mathcal{L}_\theta^i\big(\Gamma^{j+1}\big)\underline{\mathcal{F}}_{(k)(l)(k+l)}(\psi^*).$$

We denote the RHS by Err_{ml}. We have the bound:

$$\big\|v^{1+m+l}Err_{ml}\big\|^2_{L^1_u([-v\underline{v},u))L^2(S^n)} \lesssim$$

$$\lesssim \underline{v}^{1-100p}\sup_{\mathring{u}\in[-v\underline{v},u]}|\mathring{u}|^{100q}\int_{-v\underline{v}}^{\mathring{u}}\int_{S^n}v^{3+2m+2l-100p}|\hat{u}|^{100p-100q}\big|Err_{ml}\big|^2d\hat{u}.$$

The first and fifth error terms were already bounded in the proof of Proposition 3.5. The fourth error term was already bounded in the the proof of Proposition 3.4. As in the proof of Proposition 3.6, for the second error term we have the bound:

$$\sum_{i+j+k=m-1}\underline{v}^{1-100p}\sup_{\mathring{u}\in[-v\underline{v},u]}|\mathring{u}|^{100q}\int_{-v\underline{v}}^{\mathring{u}}\int_S v^{3+2k+2l-100p}|\hat{u}|^{100p-100q}\big|\nabla^k\nabla_3^l\underline{\alpha}\big|^2v^{2+2i+2j}\big|\mathcal{L}_\theta^i\big(\Gamma^{j+1}\big)\big|^2$$

$$\lesssim \underline{v}^{1-100p}\sup_{\mathring{u}\in[-v\underline{v},u]}|\mathring{u}|^{100q}\int_{-v\underline{v}}^{\mathring{u}}\sum_{(k,l)\in L}v^{3+2l-100p}|\hat{u}|^{100p-100q}\big\|\nabla_3^l\underline{\alpha}\big\|^2_{\widetilde{H}^k}\prod_{a=1}^{j+1}\sum_{(k_a,0)\in L}\int_S v^{2+2k_a}\big|\mathcal{L}_\theta^{k_a}\Gamma\big|^2,$$

which is bounded using the previous section and (3.47). Similarly, the third term is bounded by:

$$\sum_{(k,l)\in L}\underline{v}^{1-100p}\sup_{\mathring{u}\in[-v\underline{v},u]}|\mathring{u}|^{100q}\int_{-v\underline{v}}^{\mathring{u}}\int_{S^n}v^{3+2k+2l-100p}|\hat{u}|^{100p-100q}\big|\underline{\mathcal{F}}_{(k)(l-1)(k+l-1)}(\underline{\alpha})\big|^2d\mathring{V}old\hat{u},$$

which is bounded as the fourth error term. Finally, the last error term can be similarly reduced to the fifth error term. As a result, we obtain that $v^{1+m+l}\mathcal{L}_\theta^m\mathcal{L}_3^{l+1}\not{g} \in W_u^{1,1}\big([-v\underline{v},0]\big)L^2(S^n)$ and that (4.8) holds. Using self-similarity, we extend these results from the region $\{v\leq\underline{v}\}$ to all $v>0$. By continuity, we also obtain that the self-similarity relations hold along $u=0$. $\square$

Next, we compute the induced obstruction tensor $\underline{\mathcal{O}}$ and the remaining component of the scattering data $\underline{h}$:

Proposition 4.2. *There exist symmetric traceless 2-tensors* $\underline{\mathcal{O}}$ *and* $\underline{h}$*, which are independent of* u *and* v *such that for any* $0 \leq m \leq N' - 3$ *and* $\{-\underline{v}v \leq u \leq 0\}$:

$$\left\| v^{2+m} \nabla^m \underline{\mathcal{O}} \right\|_{L^2(S^n)} \lesssim \epsilon', \tag{4.10}$$

$$\left\| v^{2+m} \nabla^m \underline{h} \right\|_{L^2(S^n)} \lesssim \epsilon', \tag{4.11}$$

$$v^{2+m} \left\| v^{\frac{n-4}{2}} \nabla^m \nabla_3^{\frac{n-4}{2}} \underline{\alpha} - \nabla^m \underline{\mathcal{O}} \log \left| \frac{u}{v} \right| - \nabla^m \underline{h} \right\|_{L^2(S^n)} \lesssim \epsilon' \cdot \left| \frac{u}{v} \right|^{1-p}. \tag{4.12}$$

Proof. Step 1. Computing the induced obstruction tensor $\underline{\mathcal{O}}$*.* As usual, we first restrict to the region $\{v \leq \underline{v}\}$ to prove the desired result, then we use self-similarity to extend to all $v > 0$. We use self-similarity as in Lemma 2.2 to rewrite the Bianchi equation (3.28) for $\underline{\alpha}$ as:

$$-u\nabla_3\underline{\alpha} + \frac{n-4}{2}\underline{\alpha} - \frac{u}{2}\mathrm{tr}\underline{\chi}\underline{\alpha} = v\nabla\underline{\Psi}^G + v\psi\underline{\Psi}^G. \tag{4.13}$$

We set $l = \frac{n-4}{2}$ for the remaining of the proof. We commute (4.13) to obtain:

$$|u|\nabla_3\nabla_3^l\underline{\alpha} = v\nabla\nabla_3^l\underline{\Psi}^G + v\cdot\underline{\mathcal{F}}_{(0)(l)(l)}(\underline{\Psi}^G) + v\cdot\underline{\mathcal{F}}_{(1)(l-1)(l)}(\underline{\Psi}^G) + \underline{\mathcal{F}}_{(0)(l-1)(l-1)}(\Psi) + |u|\underline{\mathcal{F}}_{(0)(l)(l)}(\Psi).$$

We notice that in the proof of Proposition 4.1, we also obtain the bound:

$$\left\| uv^{1+m+l}\nabla^m\nabla_3^l\underline{\alpha} \right\|_{L^2(S^n)} \lesssim \epsilon' \cdot |u|^{1-p}v^{-1+p}. \tag{4.14}$$

Thus, Proposition 4.1 implies that each term on the RHS of the above equation is in $C_u^0\big([-\underline{v}v, 0]\big)\widetilde{H}^m(S^n)$. As a result, $-uv^l\nabla_3\nabla_3^l\underline{\alpha}$ can be extended to $\{u = 0\}$ as a symmetric traceless 2-tensor which is independent of the v coordinate. We define:

$$\underline{\mathcal{O}}_{AB} = \big(uv^l\nabla_3\nabla_3^l\underline{\alpha}_{AB}\big)\big|_{u=0} \in \widetilde{H}^m(S^n). \tag{4.15}$$

For any $0 \leq m \leq N' - 3$, we have the equation:

$$\nabla_3\nabla^m\nabla_3^l\underline{\alpha} = \frac{v}{|u|}\nabla^{m+1}\nabla_3^l\underline{\Psi}^G + \frac{v}{|u|}\cdot\underline{\mathcal{F}}_{(m)(l)(m+l)}(\underline{\Psi}^G) + \frac{v}{|u|}\cdot\underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\underline{\Psi}^G) +$$
$$+\frac{1}{|u|}\cdot\underline{\mathcal{F}}_{(m)(l-1)(m+l-1)}(\Psi) + \underline{\mathcal{F}}_{(m)(l)(m+l)}(\Psi).$$

Since $\lim_{u\to 0^-} u[\nabla_3, \nabla^m]\nabla_3^l\underline{\alpha} = 0$, we also obtain that:

$$\nabla^m\underline{\mathcal{O}}_{AB} = \big(uv^l\nabla_3\nabla^m\nabla_3^l\underline{\alpha}_{AB}\big)\big|_{u=0}. \tag{4.16}$$

Thus, we have the schematic equation:

$$-\nabla^m\underline{\mathcal{O}} = v^{l+1}\nabla^{m+1}\nabla_3^l\underline{\Psi}^G\big|_{u=0} + v^{l+1}\underline{\mathcal{F}}_{(m)(l)(m+l)}(\underline{\Psi}^G)\big|_{u=0} + v^{l+1}\underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\underline{\Psi}^G)\big|_{u=0}+$$

$$+v^l\underline{\mathcal{F}}_{(m)(l-1)(m+l-1)}(\Psi)\big|_{u=0}.$$

We use the bounds in Proposition 4.1, together with the bounds in the proofs of Propositions 3.4 and 3.5, to control the RHS and get:

$$\big\|v^{2+m}\nabla^m\underline{\mathcal{O}}\big\|_{L^2(S^n)} \lesssim \underline{v}^{\frac{1}{2}-p}\cdot\epsilon'.$$

Step 2. Computing the induced component of the scattering data $\underline{h}$. Extending $\nabla^m\underline{\mathcal{O}}$ to be independent of u and v, we can write the equation for $\nabla^m\nabla_3^l\underline{\alpha}$ as:

$$\partial_u\big(v^l\nabla^m\nabla_3^l\underline{\alpha}\big) = \frac{1}{|u|}\mathcal{E}_1 + \mathcal{E}_2 = \frac{1}{u}\nabla^m\underline{\mathcal{O}} + \frac{1}{|u|}\Big(\mathcal{E}_1 - \mathcal{E}_1\big|_{u=0}\Big) + \mathcal{E}_2, \tag{4.17}$$

$$\mathcal{E}_1 = v^{l+1}\nabla^{m+1}\nabla_3^l\underline{\Psi}^G + v^{l+1}\underline{\mathcal{F}}_{(m)(l)(m+l)}(\underline{\Psi}^G)+$$

$$+v^{l+1}\underline{\mathcal{F}}_{(m+1)(l-1)(m+l)}(\underline{\Psi}^G) + v^l\underline{\mathcal{F}}_{(m)(l-1)(m+l-1)}(\Psi),$$

$$\mathcal{E}_2 = v^l\underline{\mathcal{F}}_{(m)(l)(m+l)}(\Psi).$$

The key part of our proof is to establish the claim that:

$$\mathcal{E} := \frac{1}{|u|}\Big(\mathcal{E}_1 - \mathcal{E}_1\big|_{u=0}\Big) + \mathcal{E}_2 \in L^1_u\big([-\underline{v}v, 0]\big)L^2(S^n). \tag{4.18}$$

Step 2a. The proof of claim (4.18). In the proof of Proposition 3.4 we already bounded:

$$\int_{-\underline{v}v}^0 \big\|\mathcal{E}_2\big\|_{L^2(S^n)}d\hat{u} \lesssim v^{-2-m}\underline{v}^{\frac{1}{2}}\epsilon'.$$

We also have the bound:

$$\big\|\mathcal{E}_1 - \mathcal{E}_1\big|_{u=0}\big\|_{L^2(S_{u,v})} \lesssim \int_u^0 v^{l+1}\big\|\nabla_3\nabla^{m+1}\nabla_3^l\underline{\Psi}^G\big\|_{L^2(S^n)} + v^{l+1}\big\|\underline{\mathcal{F}}_{(m)(l+1)(m+l+1)}(\underline{\Psi}^G)\big\|_{L^2(S^n)}d\hat{u}+$$

$$+\int_u^0 v^{l+1}\big\|\underline{\mathcal{F}}_{(m+1)(l)(m+l+1)}(\underline{\Psi}^G)\big\|_{L^2(S^n)} + v^l\big\|\underline{\mathcal{F}}_{(m)(l)(m+l)}(\Psi)\big\|_{L^2(S^n)}d\hat{u}.$$

Using the estimates proved in region III, we have the bound for any $(i,j)\in L$:

$$\big\|\underline{\mathcal{F}}_{(i)(j)(i+j)}(\Psi)\big\|_{L^2(S^n)} \lesssim v^{-3-i-j+p}|u|^{-p}\epsilon'. \tag{4.19}$$

Since $(m,l),(m+1,l)\in L$, we have the bound:

$$\int_u^0 v^{l+1}\big\|\underline{\mathcal{F}}_{(m+1)(l)(m+l+1)}(\underline{\Psi}^G)\big\|_{L^2(S^n)} + v^l\big\|\underline{\mathcal{F}}_{(m)(l)(m+l)}(\Psi)\big\|_{L^2(S^n)}d\hat{u} \lesssim \epsilon'\cdot v^{-2-m}\left|\frac{u}{v}\right|^{1-p}.$$

We have the schematic equation as in the proof of Proposition 3.4:

$$\nabla_3 v^{1+l}\nabla^{m+1}\nabla_3^l\underline{\Psi}^G = v^{1+l}\nabla^{m+2}\nabla_3^l\Psi + v^{1+l}\underline{\mathcal{F}}_{(m+1)(l)(m+l+1)}(\Psi) + v^{1+l}\underline{\mathcal{F}}_{(m+2)(l-1)(m+l+1)}(\Psi).$$

Since $(m+2,l)\in L$, we use Proposition 3.8 and (4.19) to get:

$$\int_u^0 v^{l+1}\big\|\nabla_3\nabla^{m+1}\nabla_3^l\underline{\Psi}^G\big\|_{L^2(S^n)}d\hat{u} \lesssim \epsilon'\cdot v^{-2-m}\left|\frac{u}{v}\right|^{1-p}.$$

Using the Bianchi equations in Proposition 2.6, we can rewrite the error term:

$$\underline{\mathcal{F}}_{(m)(l+1)(m+l+1)}(\underline{\Psi}^G) = \underline{\mathcal{F}}_{(m+1)(l)(m+l+1)}(\Psi) + \sum_{\substack{i+j+k\leq m+l+1\\ i\leq l+1, k\leq m}}\nabla^k\big(\underline{\Psi}^G\nabla_3^i\psi^{j+1}\big).$$

The estimates in Proposition 3.4 and Proposition 3.5 imply the bound:

$$\sum_{\substack{i+j+|k|\leq m+l+1\\ i\leq l,|k|\leq m}}\Big\|v^{2+k_1}\nabla^{k_1}\underline{\Psi}^G\cdot v^{i+j+k_2+1}\nabla^{k_2}\nabla_3^i\psi^{j+1}\Big\|_{L^2(S^n)}\lesssim\epsilon'. \tag{4.20}$$

Similarly, we use the schematic equations (3.42) for $\nabla_3\psi$ from Proposition 3.5 to get:

$$\sum_{j+|k|\leq m}\Big\|v^{2+k_1}\nabla^{k_1}\underline{\Psi}^G\cdot v^{l+1+j+k_2+1}\nabla^{k_2}\nabla_3^{l+1}\psi^{j+1}\Big\|_{L^2(S^n)}\lesssim\epsilon'+\epsilon'\sum_{j+k\leq m}\Big\|v^{l+k+2}\nabla^k\nabla_3^{l+1}\psi\Big\|_{L^2(S^n)}$$

$$\lesssim\epsilon'+\epsilon'\sum_{k\leq m}\Big\|v^{l+k+2}\nabla^k\nabla_3^l\big(\Psi+\psi\psi\big)\Big\|_{L^2(S^n)}\lesssim\epsilon'|u|^{-p}v^p.$$

As a result, we obtain that the last remaining error term satisfies:

$$\int_u^0 v^{l+1}\big\|\underline{\mathcal{F}}_{(m)(l+1)(m+l+1)}(\underline{\Psi}^G)\big\|_{L^2(S^n)}d\hat{u}\lesssim\epsilon'\cdot v^{-2-m}\left|\frac{u}{v}\right|^{1-p}. \tag{4.21}$$

We proved that:

$$\frac{1}{|u|}\cdot\big\|\mathcal{E}_1-\mathcal{E}_1\big|_{u=0}\big\|_{L^2(S_{u,v})}\lesssim\epsilon'\cdot v^{-3-m}\left|\frac{u}{v}\right|^{-p}, \tag{4.22}$$

which proves our claim (4.18) that $\mathcal{E}\in L^1_u\big([-\underline{v}v,0]\big)L^2(S^n)$.

Step 2b. The proof of (4.11) *and* (4.12). We integrate (4.17) from $-\underline{v}v$ to u:

$$v^l\nabla^m\nabla_3^l\underline{\alpha}-\nabla^m\underline{\mathcal{O}}\log\left|\frac{u}{v}\right| = v^l\nabla^m\nabla_3^l\underline{\alpha}\big|_{u=-\underline{v}v}-\nabla^m\underline{\mathcal{O}}\log\underline{v}+\int_{-\underline{v}v}^u\mathcal{E}d\hat{u}.$$

In particular, we obtain that:

$$v^l\nabla^m\nabla_3^l\underline{\alpha}-\nabla^m\underline{\mathcal{O}}\log\left|\frac{u}{v}\right|\in W^{1,1}_u\big([-\underline{v}v,0]\big)L^2(S^n). \tag{4.23}$$

We can define the symmetric traceless two tensor $\underline{h}$ which is independent of u and v by:

$$\nabla^m\underline{h}:=\lim_{u\to0^-}\left(v^l\nabla^m\nabla_3^l\underline{\alpha}-\nabla^m\underline{\mathcal{O}}\log\left|\frac{u}{v}\right|\right). \tag{4.24}$$

The above estimates imply that:

$$\left\| v^{2+m}\nabla^m\underline{h}\right\|_{L^2(S^n)} \lesssim \epsilon'.$$

Finally, we have that:

$$v^{2+m}\left\| v^l\nabla^m\nabla_3^l\underline{\alpha} - \nabla^m\underline{\mathcal{O}}\log\left|\frac{u}{v}\right| - \nabla^m\underline{h}\right\|_{L^2(S^n)} \lesssim \int_u^0 v^{2+m}\left\|\mathcal{E}\right\|_{L^2(S^n)}d\hat{u} \lesssim \epsilon'\cdot\left|\frac{u}{v}\right|^{1-p}.$$

□

We can use the results proved so far in order to complete the proof of Theorem 4.1:

Proof of Theorem 4.1. Based on Proposition 4.1 and 4.2, we define $\slashed{g}$ and $\underline{h}$ on $\{u=0\}$. We prove that the spacetime $\big(\mathcal{M},g\big)$ satisfies the required conditions needed in order to apply the main result of [RSR18], which shows that the spacetime is determined by the induced asymptotic data $\underline{\slashed{g}_0}=\slashed{g}\big|_{(u,v)=(0,1)}$, $\underline{h}$. For N' large enough, we have:

- For any $0\leq l\leq\frac{n-2}{2}$, $0\leq m\leq N'$, the limit $\lim_{u\to0^-}\mathcal{L}_\theta^m\mathcal{L}_u^l\slashed{g}$ exists and is uniformly bounded (with appropriate self-similar v weights) by Proposition 4.1. Moreover, we remark that Proposition 4.1 also implies that we can extend to $\{u=0\}$ the following equations: the constraint equations, the ∇_4 null structure equations, the ∇_4 Bianchi equations for $\underline{\Psi}^G$ when commuted with up to $\frac{n-4}{2}$ ∇_3 derivatives; the ∇_3 null structure equations, the ∇_3 Bianchi equations, the $\nabla_4\underline{\alpha}$ Bianchi equation when commuted with up to $\frac{n-6}{2}$ ∇_3 derivatives. Using these equations on $\{u=0\}$, the argument in [RSR18, Proposition 4.3] implies that for $0<l\leq\frac{n-2}{2}$ the limits $\mathcal{L}_\theta^m\mathcal{L}_u^l\slashed{g}\big|_{(u,v)=(0,1)}$ have certain prescribed values in terms of $\underline{\slashed{g}_0}$ and satisfy the compatibility conditions.

- Proposition 4.1 implies that for any $0\leq l\leq\frac{n-6}{2}$ and $0\leq m\leq N'$, we have on $\{v=1\}$:

  $$\left\|\nabla^m\nabla_3^l\underline{\alpha}-\big(\nabla^m\nabla_3^l\underline{\alpha}\big)\big|_{u=0}\right\|_{L^2(S^n)}\lesssim\epsilon'\cdot|u|^{1-p}. \tag{4.25}$$

  We obtain a similar result for any $0\leq l\leq\frac{n-2}{2}$ and $0\leq m\leq N'$, on $\{v=1\}$:

  $$\left\|\mathcal{L}_\theta^m\mathcal{L}_3^l\hat{\slashed{g}}-\mathcal{L}_\theta^m\mathcal{L}_3^l\hat{\underline{\slashed{g}_0}}\right\|_{L^2(S^n)}\lesssim\epsilon'\cdot|u|^{1-p}. \tag{4.26}$$

- In Proposition 4.2 we define the obstruction tensor $\underline{\mathcal{O}}$ in terms of $\underline{\slashed{g}_0}$ (since all the curvature components and Ricci coefficients appearing in our definition of $\underline{\mathcal{O}}$ can be expressed in terms of $\underline{\slashed{g}_0}$). We remark that by construction $\underline{\mathcal{O}}$ satisfies the compatibility condition in [RSR18, Proposition 4.3]. Moreover, we proved that for any $0 \leq m \leq N'$ the limit:

$$\nabla^m \underline{h} = \lim_{u \to 0^-} \left(v^{\frac{n-4}{2}} \nabla^m \nabla_3^{\frac{n-4}{2}} \underline{\alpha} - \nabla^m \underline{\mathcal{O}} \log \left| \frac{u}{v} \right| \right) \tag{4.27}$$

exists and is uniformly bounded (with appropriate self-similar v weights).

- Proposition 4.2 implies that for any $0 \leq m \leq N'$, we have on $\{v = 1\}$:

$$\left\| \nabla^m \nabla_3^{\frac{n-4}{2}} \underline{\alpha} - \nabla^m \underline{\mathcal{O}} \log |u| - \nabla^m \underline{h} \right\|_{L^2(S^n)} \lesssim \epsilon' \cdot |u|^{1-p}. \tag{4.28}$$

As a result, we obtain that on $v = 1$ and $-\underline{v} \leq u \leq 0$, the metric $\slashed{g}$ induces a 1-parameter family $\hat{\slashed{g}}(u)$ of conformal classes of metrics on S^n admissible relative to $\underline{\slashed{g}_0}$, according to the definitions of [RSR18, Definition 1.2]. Thus, we can apply [RSR18, Theorem 1.1] to obtain that $\big(\mathcal{M}, g\big)$ is the unique self-similar solution with data $\big(\underline{\slashed{g}_0}, \underline{h}\big)$. Since we already know that $\big(\mathcal{M}, g\big)$ is a straight spacetime, we obtain that $\underline{h}$ must satisfy the straightness condition. Moreover, the estimates for $\underline{\slashed{g}_0}^*, \underline{\mathcal{O}}$, and $\underline{h}$ were already proved in Proposition 4.1 and 4.2.

Finally, we establish the propagation of regularity statement. Continuing the argument in the proof of Theorem 3.2, for any $K > N_3$ we can use the bounds previously obtained to repeat the proofs of Propositions 4.1 and 4.2, with $(N', q, p, \underline{v})$ replaced by $(K, q, p, \underline{v}_K)$, and the ϵ' on the right hand side of the estimates replaced by $C(K)$. As a result, we obtain that $\underline{\slashed{g}_0}$, $\underline{\mathcal{O}}$, $\underline{h} \in H^{K-3}(S^n)$ for all $K > N'$. We conclude that the straight data induced at $(u, v) = (0, 1)$ given by $\big(\underline{\slashed{g}_0}, \underline{h}\big)$ is smooth. $\square$

Chapter 3

Systems of Wave Equations on Asymptotically de Sitter Vacuum Spacetimes

5 The Model Systems

We derive the systems of commuted Bianchi equations along $\{u=-1\}$ required for the analysis of the scattering map. We also introduce the model systems, given by the principal part of the commuted Bianchi equations at top order with a general inhomogeneous term. The systems that we consider consist of wave equations that are singular at $\{v=0\}$. This section is based on [Cic24, Section 5].

Notation convention. We consider $M>N$ large enough, where $N>0$ is as in Theorem 3.1. Unless otherwise noted, for the rest of the thesis we write $A\lesssim B$ for some quantities $A, B>0$ if there exists a constant $C>0$ depending only on M such that $A\leq CB$.

Integration convention. For the remainder of the thesis we make the convention that the volume form used on the sphere $S_{u,v}=\{(u,v)\}\times S^n$ with induced metric $\slashed{g}_{u,v}$ is $d\slashed{Vol}_{\slashed{g}_{u,v}}$, in order to be consistent with the notation in [Cic26]. We note that this convention is different from the one in Section 3 and Section 4.

5.1 Bianchi Equations

We write the Bianchi system along $\{u = -1\}$ as a system of wave equations. Using the facts that $\nabla_S \Psi = -2\Psi$ and $\mathrm{tr}\underline{\chi} = v\mathrm{tr}\chi - n$ (according to Lemma 2.2 and (2.18)), we can rewrite the equations in Proposition 2.6 for the Bianchi pairs on $\{u = -1\}$ as follows:

$$v\nabla_4 \alpha_{AB} + \Big(2 - \frac{n}{2} + \frac{v}{2}\mathrm{tr}\chi\Big)\alpha_{AB} = -\nabla^C \nu_{C(AB)} + \mathcal{E}_1^{(3)} \tag{5.1}$$

$$\nabla_4 \nu_{ABC} = -2\nabla_{[A}\alpha_{B]C} + \mathcal{E}_{1/2}^{(4)} \tag{5.2}$$

$$v\nabla_4 \nu_{ABC} + \frac{2v}{n}\mathrm{tr}\chi \nu_{ABC} = -2\nabla_{[A}\tau_{B]C} + 2\hat{\underline{\chi}}_{[A}^D \nu_{|D|B]C} + \mathcal{E}_{3/2}^{(3)} \tag{5.3}$$

$$\nabla_4 R_{ABCD} = -2\nabla_{[A}\nu_{|CD|B]} + \mathcal{E}_1^{(4)} \tag{5.4}$$

$$\begin{aligned} v\nabla_4 R_{ABCD} + \frac{2v}{n}\mathrm{tr}\chi R_{ABCD} &= -2\nabla_{[A}\underline{\nu}_{|CD|B]} + \underline{\chi}_{A[D}\tau_{C]B} + \underline{\chi}_{B[C}\tau_{D]A} \\ &\quad + 2\hat{\underline{\chi}}_{[A}^E R_{B]ECD} + \mathcal{E}_2^{(3)} \end{aligned} \tag{5.5}$$

$$\nabla_4 \underline{\nu}_{ABC} = -2\nabla_{[A}\tau_{B]C} + \mathcal{E}_{3/2}^{(4)} \tag{5.6}$$

$$v\nabla_4 \underline{\nu}_{ABC} + \Big(-1 + \frac{3v}{n}\mathrm{tr}\chi\Big)\underline{\nu}_{ABC} = -2\nabla_{[A}\underline{\alpha}_{B]C} + 2\hat{\underline{\chi}}_{[A}^D \underline{\nu}_{B]DC} + 2\hat{\underline{\chi}}_{[A}^D \underline{\nu}_{|CD|B]} + \mathcal{E}_{5/2}^{(3)} \tag{5.7}$$

$$\nabla_4 \underline{\alpha}_{AB} = -\nabla^C \underline{\nu}_{C(AB)} + \mathcal{E}_2^{(4)}. \tag{5.8}$$

Using the commutation formulas in Lemma 2.7, we can rewrite the system of Bianchi equations on $\{u = -1\}$ as a system of wave equations:

Proposition 5.1. *We have the system of wave equations on $\{u = -1\}$ for any $0 \le m \le M$, $0 \le l \le \frac{n}{2} - 2$:*

$$\begin{cases} v\nabla_4^2 \nabla^m \nabla_4^l \alpha + \big(3 + l - \frac{n}{2}\big)\nabla_4 \nabla^m \nabla_4^l \alpha - \Delta \nabla^m \nabla_4^l \alpha = \psi \nabla^{m+1}\nabla_4^l \Psi + Err_{ml}^{\Psi} \\ v\nabla_4^2 \nabla^m \nabla_4^l \Psi^G + \big(2 + l - \frac{n}{2}\big)\nabla_4 \nabla^m \nabla_4^l \Psi^G - \Delta \nabla^m \nabla_4^l \Psi^G = \sum_{\Psi_0^G} \psi \nabla^{m+1}\nabla_4^l \Psi_0^G + Err_{ml}^{\Psi}, \end{cases} \tag{5.9}$$

where we have the error term notation:

$$\begin{aligned} Err_{ml}^{\Psi} = {} & v\mathcal{F}_{(m)(l+1)(l+m+1)}(\Psi) + v\mathcal{F}_{(m+1)(l)(l+m+1)}(\Psi) + \mathcal{F}_{(2+m)(l-1)(l+m+1)}(\Psi) + \\ & + \mathcal{F}_{(m)(l)(m+l)}(\Psi) + \mathcal{F}^{lot}_{(m+1)(l)(m+l+1)}(\Psi) + \nabla^m \nabla_4^l \big(\Psi \cdot \Psi^G\big) + \\ & + \sum_{i+2j=m} \nabla^i \big(Ri\not{e}m^{j+1} \cdot \nabla_4^l \Psi\big) + \nabla^m \nabla_4^l \nabla\big(\psi\psi\big) + \nabla^m \nabla_4^l \nabla\big(\psi\psi\psi\big) \end{aligned}$$

and we point out that in the RHS of (5.9) *we sum the terms for all* Ψ. *Similarly, using the schematic equation:*

$$\nabla_4 \nabla^m \nabla_4^l \Psi^G = \psi \nabla^{m+1} \nabla_4^l \Psi + Err_{ml}^{\Psi}, \tag{5.10}$$

we also have the system of wave equations on $\{u=-1\}$ *for any* $0 \leq m \leq M$, $0 \leq l \leq \frac{n}{2}-2$:

$$\begin{cases} v\nabla_4^2 \nabla^m \nabla_4^l \alpha + \Big(3+l-\frac{n}{2}\Big) \nabla_4 \nabla^m \nabla_4^l \alpha - \Delta \nabla^m \nabla_4^l \alpha = \psi \nabla^{m+1} \nabla_4^l \Psi + Err_{ml}^{\Psi} \\ v\nabla_4^2 \nabla^m \nabla_4^l \Psi^G + \Big(3+l-\frac{n}{2}\Big) \nabla_4 \nabla^m \nabla_4^l \Psi^G - \Delta \nabla^m \nabla_4^l \Psi^G = \psi \nabla^{m+1} \nabla_4^l \Psi + Err_{ml}^{\Psi}. \end{cases} \tag{5.11}$$

Remark 5.1. *We rewrite the system of Bianchi equations in two ways since for the good curvature components* Ψ^G *we have flexibility on which equations to use when proving estimates. In particular,* (5.11) *is easier to use when evolving forward from* $\{v=0\}$*, whereas it is convenient to use* (5.9) *when evolving towards* $\{v=0\}$. *On the other hand, we notice that for the curvature component* α *we always work with the same equation.*

Proof. Using equations (5.1)-(5.2) satisfied by the Bianchi pair (α, ν), we obtain:

$$v\nabla_4^2 \alpha_{AB} + \left(3-\frac{n}{2}\right)\nabla_4 \alpha_{AB} = -\nabla_4 \nabla^C \nu_{C(AB)} + \nabla_4\big(\psi \Psi^G\big) + \nabla_4\big(v\psi\Psi\big) =$$

$$= \Delta \alpha_{AB} + Ri\not{e}m \cdot \alpha + \nabla_4\big(\psi\Psi^G\big) + \nabla_4\big(v\psi\Psi\big) + \nabla\big(\psi\Psi^G\big) =$$

$$= \Delta\alpha_{AB} + \psi\nabla\Psi + \Psi\Psi^G + \mathcal{F}_{101}^{lot}(\Psi) + v\mathcal{F}_{011}(\Psi),$$

where we also used the other Bianchi equations in Proposition 2.6 and the null structure equations in Proposition 2.4 on $\{u=-1\}$. We commute with ∇_4^l:

$$v\nabla_4^2 \nabla_4^l \alpha_{AB} + \left(3+l-\frac{n}{2}\right)\nabla_4 \nabla_4^l \alpha_{AB} - \Delta \nabla_4^l \alpha_{AB} = \psi \nabla \nabla_4^l \Psi + \mathcal{F}_{(2)(l-1)(l+1)}(\Psi)$$

$$+ \mathcal{F}^{lot}_{(1)(l)(l+1)}(\Psi) + v\mathcal{F}_{(0)(l+1)(l+1)}(\Psi) + \nabla_4^l\big(\Psi\Psi^G\big).$$

Commuting with ∇^m, we obtain the equation for $\nabla^m \nabla_4^l \alpha$. Next, we compute the wave equation satisfied by ν:

$$v\nabla_4^2 \nu_{ABC} = -2\nabla_{[A} v \nabla_4 \alpha_{B]C} + v\nabla\big(\psi\Psi\big) + v\nabla_4\big(\psi\Psi^G\big) =$$

$$= \nabla_A \nabla^D \nu_{D(BC)} - \nabla_B \nabla^D \nu_{D(AC)} - \left(2-\frac{n}{2}\right)\nabla_4 \nu_{ABC} + \mathcal{F}_{101}(\Psi^G) + v\mathcal{F}_{101}(\Psi) + v\mathcal{F}_{011}(\Psi).$$

Using the constraint equations $\nu_{ABC} = 2\nabla_{[A}\chi_{B]C}$ and $\nabla^A\chi_{AB} = \nabla_B \mathrm{tr}\chi$ in Proposition 2.5, we can rewrite the first two terms on the RHS as $\Delta\nu_{ABC} + \nabla\big(Ri\cancel{e}m \cdot \chi\big)$. As a result, we get:

$$v\nabla_4^2\nu_{ABC} = \Delta\nu_{ABC} - \left(2 - \frac{n}{2}\right)\nabla_4\nu_{ABC} + \psi\nabla\Psi^G + \mathcal{F}_{101}^{lot}(\Psi) + v\mathcal{F}_{101}(\Psi) + v\mathcal{F}_{011}(\Psi) + \nabla(\psi\psi\psi).$$

We remark that the term $\psi\nabla\alpha$ does not appear on the RHS. We commute as before to get the equation for $\nabla^m\nabla_4^l\nu$, which does not contain $\psi\nabla^{m+1}\nabla_4^l\alpha$ on the RHS.

We use a similar argument for the remaining curvature components, and we briefly note here the structure of the equations that we use to rule out the dangerous term $\psi\nabla^{m+1}\nabla_4^l\alpha$ on the RHS. For R we consider (5.5) and note by signature considerations that α, ν are absent. We differentiate the equation in v and use the fact that (5.6),(5.8) do not contain α, while (5.4) does not contain $\nabla\alpha$. For ν we use the same argument, starting with (5.7). We use a similar argument for $\underline{\alpha}$, starting with (5.8) and using the fact that α is absent in (5.7). Thus, we proved (5.9).

Finally, we note that (5.10) follows from the commutation formulas in Lemma 2.7. Thus, using this in (5.9) we obtain (5.11) as well. $\square$

5.2 Model Systems

In this section we introduce two systems of linear wave equations on the background spacetime obtained in Theorem 3.1. These systems will model the linear part of the commuted Bianchi equations on $\{u = -1\}$ from the previous section, with the nonlinear part being contained in an inhomogeneous term. In Section 7 and Section 8, we provide a detailed study of solutions to the model systems.

As inspired by our treatment of the linear wave equation on the background of de Sitter space in [Cic23], we consider the new time variable[1]:

$$\tau = \sqrt{v},\ e_4 = \frac{1}{2\tau}\partial_\tau.$$

We recall that all the tensors along $\{u = -1\}$ are expressed in a Lie propagated frame with respect to $e_4 = \partial_v$. We notice that $\mathcal{L}_4 e_A = 0$ is equivalent to $\mathcal{L}_\tau e_A = 0$, so we can extend any tensors defined only at $\{\tau = 0\}$ to be independent of τ. For any horizontal k-tensor Φ we compute that:

$$\nabla_\tau\Phi = \mathcal{L}_\tau\Phi + \tau\chi \cdot \Phi \tag{5.12}$$

---

[1] We point out that the time variable τ should not be confused with the curvature component τ_{AB} introduced in Section 2.1.

$$\nabla_\tau \nabla_\tau \Phi_{A_1 \dots A_k} = \nabla_\tau \big(\nabla_\tau \Phi \big)_{A_1 \dots A_k} - \frac{1}{\tau} \nabla_\tau \Phi_{A_1 \dots A_k}.$$

We consider the following model system:

Definition 5.1 (First Model System)**.** *We assume that the smooth horizontal tensors* $\Phi_0, \dots, \Phi_I$ *defined on the hypersurface* $\{u = -1\} \times \{\tau \in (0,1)\} \times S^n$ *of the spacetime* $\big(\mathcal{M}, g\big)$ *obtained in Theorem 3.1 satisfy the expansions for all* $1 \leq i \leq I$*:*

$$\Phi_0 = 2\mathcal{O} \log \tau + h + O\big(\tau^2 |\log \tau|^2\big),\ \nabla_\tau \Phi_0 = \frac{2\mathcal{O}}{\tau} + O\big(\tau |\log \tau|^2\big) \ \textit{in } C^\infty(S^n),$$

$$\Phi_i = \Phi_i^0 + O\big(\tau^2 |\log \tau|^2\big),\ \nabla_\tau \Phi_i = O\big(\tau |\log \tau|^2\big) \ \textit{in } C^\infty(S^n),$$

and the model system of wave equations on $\{u = -1\}$ *for any* $0 \leq m \leq M$*:*

$$\begin{cases} \nabla_\tau \big(\nabla_\tau \nabla^m \Phi_0 \big) + \frac{1}{\tau} \nabla_\tau \nabla^m \Phi_0 - 4 \Delta \nabla^m \Phi_0 = \psi \nabla^{m+1} \Phi + F_m^0 \\ \nabla_\tau \big(\nabla_\tau \nabla^m \Phi_i \big) + \frac{1}{\tau} \nabla_\tau \nabla^m \Phi_i - 4 \Delta \nabla^m \Phi_i = \psi \nabla^{m+1} \Phi + F_m^i, \end{cases} \tag{5.13}$$

where the inhomogeneous terms satisfy $F_m^0, F_m^i \in L^1_\tau\big([0,1]\big) C^\infty(S^n)$*, and the covariant angular derivatives are with respect to the metric* $\slashed{g}_\tau := \slashed{g}_{u=-1, v=\tau^2}$ *induced on* $S_\tau = \{u = -1\} \times \{\tau\} \times S^n$*.*

Based on equation (5.11), we obtain that:

$$\Phi_0 = \nabla_4^{\frac{n-4}{2}} \alpha,\ \Phi_i = \nabla_4^{\frac{n-4}{2}} \Psi^G,\ F_m^0 = Err^{\Psi}_{m, \frac{n-4}{2}},\ F_m^i = Err^{\Psi}_{m, \frac{n-4}{2}}$$

satisfy the first model system, where the desired asymptotic expansions follow by Section 4, or similarly by [RSR18]. We prove estimates for this system in Section 7, and use these in Section 9 to obtain estimates for the commuted Bianchi system at finite times in terms of the asymptotic data at $\{v = 0\}$.

Similarly, we also consider the following model system:

Definition 5.2 (Second Model System)**.** *We assume that the smooth horizontal tensors* $\Phi_0, \dots, \Phi_I$ *defined on the hypersurface* $\{u = -1\} \times \{\tau \in (0,1)\} \times S^n$ *of the spacetime* $\big(\mathcal{M}, g\big)$ *obtained in Theorem 3.1 satisfy the expansions for all* $1 \leq i \leq I$*:*

$$\Phi_0 = 2\mathcal{O} \log \tau + h + O\big(\tau^2 |\log \tau|^2\big),\ \nabla_\tau \Phi_0 = \frac{2\mathcal{O}}{\tau} + O\big(\tau |\log \tau|^2\big) \ \textit{in } C^\infty(S^n),$$

$$\Phi_i = \Phi_i^0 + O\big(\tau^2 |\log \tau|^2\big),\ \nabla_\tau \Phi_i = O\big(\tau |\log \tau|^2\big) \ \textit{in } C^\infty(S^n),$$

and the model system of wave equations on $\{u=-1\}$ *for any* $0\leq m\leq M$*:*

$$\begin{cases}\nabla_\tau\big(\nabla_\tau\nabla^m\Phi_0\big)+\frac{1}{\tau}\nabla_\tau\nabla^m\Phi_0-4\Delta\nabla^m\Phi_0=\psi\nabla^{m+1}\Phi+F^0_m\\ \nabla_\tau\big(\nabla_\tau\nabla^m\Phi_i\big)-\frac{1}{\tau}\nabla_\tau\nabla^m\Phi_i-4\Delta\nabla^m\Phi_i=\sum_{j\neq 0}\psi\nabla^{m+1}\Phi_j+F^i_m,\end{cases}\tag{5.14}$$

where the inhomogeneous terms satisfy $F^0_m, F^i_m\in L^1_\tau\big([0,1]\big)C^\infty(S^n)$*, and the covariant angular derivatives are with respect to the metric* $\slashed{g}_\tau:=\slashed{g}_{u=-1,v=\tau^2}$ *induced on* $S_\tau=\{u=-1\}\times\{\tau\}\times S^n$*.*

As before, equation (5.9) implies that:

$$\Phi_0=\nabla_4^{\frac{n-4}{2}}\alpha,\ \Phi_i=\nabla_4^{\frac{n-4}{2}}\Psi^G,\ F^0_m=Err^\Psi_{m,\frac{n-4}{2}},\ F^i_m=Err^\Psi_{m,\frac{n-4}{2}}$$

satisfy the second model system. We prove estimates for this system in Section 8, and use these in Section 10 to obtain estimates for the asymptotic data at $\{v=0\}$, $\{u=0\}$ to the commuted Bianchi system in terms of initial data at finite times.

Remark 5.2. *By writing the estimates in Theorem 3.1 using the* τ *coordinate, we get that the background metric* $\slashed{g}_\tau$ *satisfies the following bounds:*

$$\sum_{i=0}^N\sup_{\mathcal{M}}\big|\mathcal{L}^i_\theta\big(\slashed{g}_\tau-\slashed{g}_{dS}\big)\big|\lesssim\epsilon,\tag{5.15}$$

$$\sup_{\tau\in(0,1]}\big\|\chi\big\|_{H^N(S_\tau)}+\sup_{\tau\in(0,1]}\big\|\nabla_\tau\chi\big\|_{H^N(S_\tau)}\lesssim 1.\tag{5.16}$$

Remark 5.3. *It is practical to introduce two model systems, since in the case of the regular quantities* Φ_i *we have flexibility on which equations to use when proving estimates. In particular,* (5.13) *is easier to use when evolving forward from* $\{\tau=0\}$*, whereas it is convenient to use* (5.14) *when evolving towards* $\{\tau=0\}$*. On the other hand, we point out that for the singular quantity* Φ_0 *we always use the same equation.*

Remark 5.4. *In [Cic26] we give an equivalent definition of the model systems using the fact that* $\slashed{g}_\tau=\widetilde{\slashed{g}}(\log(2\tau))$*, where* $\widetilde{\slashed{g}}$ *is defined in* (1.3) *and* $\tau=e^T/2$*, as in Remark 1.7. We also remark that the necessary assumptions on the background spacetime required in [Cic26] follow by* (5.15)-(5.16)*.*

6 Geometric Littlewood-Paley Theory

We introduce the geometric Littlewood-Paley theory of Klainerman-Rodnianski [KR06]. We also use the methods of [KR06, KR05] to prove additional new results that are needed in the proofs of the sharp estimates in Theorems 7.1 and 8.1, as explained in the Introduction. This section is based on [Cic26, Section 2].

6.1 Bounds for the Heat Equation

The geometric Littlewood-Paley projections are defined using the heat flow. In this section, we introduce some standard properties of the heat equation based on [KR06], and we prove additional commutation estimates. We point out that while the results stated here hold under more general assumptions, it suffices for our purposes to use the bounds (5.15) and (5.16) on the background spacetime.

For any tensor field F on S_τ, we denote by $U(z)F$ the solution on $[0,\infty) \times S_\tau$ to the heat equation:

$$\partial_z U(z)F - \Delta U(z)F = 0, \; U(0)F = F, \tag{6.1}$$

where Δ is the Laplace-Beltrami operator on $\big(S_\tau, \not{g}\big)$, for some $\tau \in [0,1]$. We notice that the operators U are self-adjoint and form a semigroup.

We use the following estimates for the heat kernel of [KR06]:

Proposition ([KR06, Proposition 4.1]). *We have the estimates for the operator $U(z)$:*

$$\begin{aligned}
\big\|U(z)F\big\|_{L^2} &\leq \|F\|_{L^2} \\
\big\|\nabla U(z)F\big\|_{L^2} &\leq \|\nabla F\|_{L^2} \\
\big\|\nabla U(z)F\big\|_{L^2} + \big\|U(z)\nabla F\big\|_{L^2} &\leq \sqrt{2} z^{-\frac{1}{2}} \|F\|_{L^2} \\
\big\|\Delta U(z)F\big\|_{L^2} &\leq \frac{\sqrt{2}}{2} z^{-1} \|F\|_{L^2}.
\end{aligned}$$

We prove additional estimates using the methods of [KR06, KR05]. In particular, we notice the importance of proving estimates for the commutation of the operator $U(z)$ and the e_4 vector field.

Lemma 6.1. *We have the estimates for the operator $U(z)$:*

$$\big\|\nabla^m U(z)F\big\|_{L^2} \lesssim C\big(\|Ri\not{e}m\|_{H^{m-2}}\big) \cdot \|F\|_{H^m} \tag{6.2}$$

$$\big\|[U(z), \nabla]F\big\|_{L^2} \lesssim (\sqrt{z} + z) \cdot \big\|F\big\|_{L^2} \tag{6.3}$$

$$\big\|\nabla[U(z),\nabla]F\big\|_{L^2} \lesssim (1+\sqrt{z})\cdot\big\|F\big\|_{L^2} \tag{6.4}$$

$$\big\|[U(z),\nabla^m]F\big\|_{L^2} \lesssim (\sqrt{z}+z)C\big(\|Ri\not{e}m\|_{H^{m-1}}\big)\cdot\big\|F\big\|_{H^{m-1}} \tag{6.5}$$

$$\big\|\nabla[U(z),\nabla^m]F\big\|_{L^2} \lesssim (\sqrt{z}+z)C\big(\|Ri\not{e}m\|_{H^{m}}\big)\cdot\big\|F\big\|_{H^{m}} \tag{6.6}$$

$$\big\|[U(z),G]F\big\|_{L^2} \lesssim (z+\sqrt{z})\big\|G\big\|_{W^{2,\infty}}\big\|F\big\|_{L^2} \tag{6.7}$$

$$\big\|\nabla[U(z),G]F\big\|_{L^2} \lesssim (1+\sqrt{z})\big\|G\big\|_{W^{2,\infty}}\big\|F\big\|_{L^2} \tag{6.8}$$

$$\big\|[U(z),\nabla_4]F\big\|_{L^2} \lesssim (1+z)\cdot\big\|F\big\|_{L^2} \tag{6.9}$$

$$\big\|\nabla[U(z),\nabla_4]F\big\|_{L^2} \lesssim (1+z)\cdot\big\|F\big\|_{H^1} \tag{6.10}$$

$$\big\|\nabla^2U(z)F\big\|_{L^2} \lesssim \big(1+z^{-3/4}\big)\cdot\big\|F\big\|_{H^{1/2}}. \tag{6.11}$$

Proof. We prove (6.2) by induction, while considering separately the cases $m \leq 2n$ and $m > 2n$. The case $m = 1$ is proved in [KR06]. We assume at first that (6.2) holds up to $m < 2n$, and prove it for $m+1$. We notice that for any tensor ϕ we have:

$$\|\nabla^2\phi\|_{L^2} \lesssim \|\Delta\phi\|_{L^2} + \|Ri\not{e}m\|_{L^\infty}\|\phi\|_{H^1}.$$

As a result, we get:

$$\|\nabla^{m+1}U(z)F\|_{L^2} \lesssim \|\Delta\nabla^{m-1}U(z)F\|_{L^2} + \|Ri\not{e}m\|_{L^\infty}\|\nabla^{m-1}U(z)F\|_{H^1}.$$

Since $\|Ri\not{e}m\|_{L^\infty} \lesssim 1$ by the bound on the background spacetime in (5.15), the second term is controlled using the induction hypothesis. For the first term, we write using $[\Delta, U(z)]F = 0$:

$$\|\Delta\nabla^{m-1}U(z)F\|_{L^2} \lesssim \|\nabla^{m-1}\Delta U(z)F\|_{L^2} + \big\|[\Delta,\nabla^{m-1}]U(z)F\big\|_{L^2}$$

$$\lesssim \|\nabla^{m-1}U(z)\Delta F\|_{L^2} + \sum_{i+2j=m-1}\big\|\nabla^i(Ri\not{e}m^{j+1}U(z)F)\big\|_{L^2} \lesssim \|F\|_{H^{m+1}},$$

by using (5.15) again. This completes the first induction argument, and establishes (6.2) for all $m \leq 2n$. Next, for some $m \geq 2n$ we assume that (6.2) holds up to m, and prove it for $m+1$. As before, we have:

$$\|\nabla^{m+1}U(z)F\|_{L^2} \lesssim \|\nabla^{m-1}U(z)\Delta F\|_{L^2} + \big\|[\Delta,\nabla^{m-1}]U(z)F\big\|_{L^2} + C\big(\|Ri\not{e}m\|_{H^{m-2}}\big)\cdot\|F\|_{H^m}$$

$$\lesssim C\big(\|Ri\not{e}m\|_{H^{m-2}}\big)\cdot\|F\|_{H^{m+1}} + \sum_{i+2j=m-1}\big\|\nabla^i(Ri\not{e}m^{j+1}U(z)F)\big\|_{L^2}.$$

The second term in the above can be written as:

$$\sum_{i+2j=m-1}\sum_{i_0+\ldots+i_{j+1}=i}\left\|\left(\nabla^{i_0}U(z)F\right)\prod_{l=1}^{j+1}\left(\nabla^{i_l}Ri\not{e}m\right)\right\|_{L^2}.$$

For each terms of the sum, we bound the factor with the most number of angular derivatives in L^2. The other factors have at most $(m-1)/2$ derivatives, so we bound them in L^∞ and apply the Sobolev inequality to get:

$$\|\nabla^{m+1}U(z)F\|_{L^2}\lesssim C\big(\|Ri\not{e}m\|_{H^{m-1}}\big)\cdot\|F\|_{H^{m+1}}.$$

This completes the proof of (6.2). Next, we prove (6.3). By Duhamel's formula as in [KR05] we have:

$$[U(z),\nabla]F=\int_0^z U(z-z')[\Delta,\nabla]U(z')F dz'.$$

Using the standard estimates for the heat kernel in [KR06], we get (6.3). The proof of (6.4) is similar. We also have using $[\Delta,\nabla^m]=\nabla[\Delta,\nabla^{m-1}]+[\Delta,\nabla]\nabla^{m-1}$:

$$[U(z),\nabla^m]F=\int_0^z U(z-z')[\Delta,\nabla^m]U(z')F dz'=$$

$$=\int_0^z U(z-z')\nabla\big(Ri\not{e}m\nabla^{m-1}+[\Delta,\nabla^{m-1}]\big)U(z')F+U(z-z')\big(\nabla Ri\not{e}m\cdot\nabla^{m-1}U(z')F\big)dz'.$$

This implies (6.5) using the bounds in (6.2), and by estimating the commutator term as in the proof of (6.2). The proof of (6.6) is similar. The proofs of (6.7) and (6.8) follow, since according to [KR06]:

$$[U(z),G]F=\int_0^z U(z-z')\big(\Delta G\cdot U(z')F+2\nabla G\cdot\nabla U(z')F\big)dz'.$$

By Duhamel's formula as in [KR05] we have:

$$[U(z),\nabla_4]F=\int_0^z U(z-z')[\Delta,\nabla_4]U(z')F dz'.$$

Using (6.13) and the standard estimates for the heat kernel in [KR06], we get (6.9). The proof of (6.10) is similar. Finally, to prove (6.11):

$$\big\|\nabla^2U(z)F\big\|_{L^2}^2\lesssim\sum_{k\in\mathbb{Z}}\big\|P_k\nabla^2U(z)F\big\|_{L^2}^2\lesssim(1+z^{-1})\big\|F\big\|_{L^2}^2+\sum_{k\geq0}\big\|\nabla^2P_kU(z)F\big\|_{L^2}^2\lesssim$$

$$\lesssim(1+z^{-1})\big\|F\big\|_{L^2}^2+\sum_{k\geq0}\big\|\nabla^2U(z)P_kF\big\|_{L^2}^2\lesssim(1+z^{-1})\big\|F\big\|_{L^2}^2+\sum_{k\geq0}z^{-1}\big\|P_kF\big\|_{L^2}\cdot z^{-1/2}\big\|\nabla P_kF\big\|_{L^2}$$

$$\lesssim(1+z^{-1})\big\|F\big\|_{L^2}^2+\sum_{k\geq0}z^{-3/2}2^k\big\|P_kF\big\|_{L^2}\big\|\underline{P}_kF\big\|_{L^2}\lesssim(1+z^{-1})\big\|F\big\|_{L^2}^2+z^{-3/2}\big\|F\big\|_{H^{1/2}}^2.$$

We note that the proof of the last statement uses parts of Lemma 6.3. We included its statement in the heat equation bounds section for future convenience. $\square$

In our proofs we will repeatedly use the commutation formulas which follow from Lemmas 2.5 and 2.7.

Lemma 6.2. *We have the commutation formulas:*

$$[\nabla, \nabla_4]\phi = \nabla\chi \cdot \phi + \chi \cdot \nabla\phi, \tag{6.12}$$

$$[\Delta, \nabla_4]\phi = \nabla(\chi\nabla\phi) + O\big(\|\chi\|_{W^{2,\infty}}(|\nabla\phi| + |\phi|)\big). \tag{6.13}$$

6.2 Bounds for the LP Projections

In this section, we follow [KR06] to define the LP projections using the heat flow. We then prove a series of additional bounds for the LP projections that are needed in our analysis.

The class $\mathcal{M}_{\text{op}}$ of smooth symbols defined in [KR06] consists of smooth functions $m : [0, \infty) \to \mathbb{R}$, which decay at infinity and satisfy the vanishing moments property:

$$\int_0^\infty z^{k_1}\partial_z^{k_2} m(z)dz = 0, \ |k_1| + |k_2| \leq K,$$

for large enough order $K > 0$. We set $m_k(z) = 2^{2k}m(2^{2k}z)$ for any $k \in \mathbb{Z}$.

Each symbol $m \in \mathcal{M}_{\text{op}}$ gives rise to an LP projection operator. For any tensor field F on S_τ, we define the LP projections corresponding to the symbol m for $k \in \mathbb{Z}$:

$$P_k F = \int_0^\infty m_k(z)U(z)F dz.$$

We refer the reader to Theorem 5.5 in [KR06] for the fundamental properties of these operators, similar to the standard LP projections. We use the following estimates for the LP projections of [KR06]:

Proposition ([KR06, Theorem 5.5, Remark 5.6])**.** *For an arbitrary LP projection, and any smooth tensor F we have:*

1. *Bessel inequality.*

$$\sum_{k\in\mathbb{Z}} \|P_k F\|_{L^2}^2 \lesssim \|F\|_{L^2}^2.$$

2. *Finite band property.*

$$\|\nabla P_k F\|_{L^2} \lesssim 2^k \|F\|_{L^2}, \ \|P_k F\|_{L^2} \lesssim 2^{-k} \|\nabla F\|_{L^2},$$

$$\|\Delta P_k F\|_{L^2} \lesssim 2^{2k} \|F\|_{L^2}, \ \|P_k F\|_{L^2} \lesssim 2^{-2k} \|\Delta F\|_{L^2}.$$

3. *L^2-almost orthogonality. For any two families of LP projections $P_k, \widetilde{P}_k$ with symbols $m, \widetilde{m} \in \mathcal{M}_{\text{op}}$ we have:*

$$\|P_k \widetilde{P}_{k'} F\|_{L^2} \lesssim 2^{-4|k-k'|} \cdot \|F\|_{L^2}.$$

Our main use of the geometric LP projections is to define fractional Sobolev spaces H^a for $a \geq 0$. These are defined in [KR06, Definition 7.9] using fractional powers of the operator $I - \Delta$. We can give an equivalent characterization of the fractional Sobolev norms according to the following result of [KR06]:

Proposition ([KR06, Corollary 7.12])**.** *For an arbitrary LP projection, $a \geq 0$ and any smooth tensor F, we have:*

$$\sum_{k \geq 0} 2^{2ak} \|P_k F\|_{L^2}^2 \lesssim \|F\|_{H^a}^2.$$

Moreover, if $\sum_k P_k^2 = I$, then:

$$\|F\|_{H^a}^2 \lesssim \sum_{k \geq 0} 2^{2ak} \|P_k F\|_{L^2}^2 + \|F\|_{L^2}^2.$$

We use this result to give an equivalent definition of fractional Sobolev spaces:

Definition 6.1. *Let P_k be a family of projections with $\sum_k P_k^2 = I$. We write any $a \geq 0$ as $a = [a] + \{a\}$, with $\{a\} \in [0, 1)$. For any smooth tensor F we define its Sobolev norm of order a as:*

$$\|F\|_{H^a}^2 := \|F\|_{H^{[a]}}^2 + \|\nabla^{[a]} F\|_{H^{\{a\}}}^2,$$

where we define the $H^{[a]}$ and $H^{\{a\}}$ Sobolev norms by:

$$\|F\|_{H^{[a]}}^2 := \sum_{i=0}^{[a]} \|\nabla^i F\|_{L^2}^2, \ \|F\|_{H^{\{a\}}}^2 := \sum_{k \geq 0} 2^{2\{a\}k} \|P_k F\|_{L^2}^2 + \|F\|_{L^2}^2.$$

Following the ideas of [KR06], we prove additional commutation bounds. As before, we highlight the importance of proving estimates for the commutation with e_4, which allow us to control the change of the projection operators in terms of the time τ. Moreover, we notice that the following bounds rely on our previous estimates for the heat flow in Lemma 6.1.

Lemma 6.3. *The LP projection operators satisfy the following bounds for $k \geq 0$:*

$$[\nabla_t, P_k] = \frac{t}{2^{2k-1}}[\nabla_4, P_k], \text{ where } t = 2^k\tau \tag{6.14}$$

$$\big\|[\nabla^m, P_k]F\big\|_{L^2} \lesssim 2^{-k}C\big(\|Ri\not{e}m\|_{H^{m-1}}\big)\cdot\big\|F\big\|_{H^{m-1}} \tag{6.15}$$

$$\big\|\nabla[\nabla^m, P_k]F\big\|_{L^2} \lesssim 2^{-k}C\big(\|Ri\not{e}m\|_{H^m}\big)\cdot\big\|F\big\|_{H^m} \tag{6.16}$$

$$\big\|[\nabla_4, P_k]F\big\|_{L^2} \lesssim \big\|F\big\|_{L^2} \tag{6.17}$$

$$\big\|\nabla[\nabla_4, P_k]F\big\|_{L^2} \lesssim \big\|F\big\|_{H^1} \tag{6.18}$$

$$\big\|[P_k, G]F\big\|_{L^2} \lesssim 2^{-k}\big\|G\big\|_{W^{2,\infty}}\big\|F\big\|_{L^2} \tag{6.19}$$

$$\big\|\nabla[P_k, G]F\big\|_{L^2} \lesssim \big\|G\big\|_{W^{2,\infty}}\big\|F\big\|_{L^2}. \tag{6.20}$$

Proof. The first formula follows since $\nabla_t = 2^{-k}\nabla_\tau = 2^{-k+1}\tau\nabla_4 = 2^{-2k+1}t\nabla_4$. In order to prove (6.15), we notice that using the bound in (6.5) we have that:

$$\big\|[\nabla^m, P_k]F\big\|_{L^2} \leq \int_0^\infty |m_k|\cdot\big\|[\nabla^m, U(z)]F\big\|_{L^2}dz \lesssim 2^{-k}C\big(\|Ri\not{e}m\|_{H^{m-1}}\big)\cdot\big\|F\big\|_{H^{m-1}}.$$

Similarly, we also have that:

$$\big\|\nabla[\nabla^m, P_k]F\big\|_{L^2} \leq \int_0^\infty |m_k|\cdot\big\|\nabla[\nabla^m, U(z)]F\big\|_{L^2}dz \lesssim 2^{-k}C\big(\|Ri\not{e}m\|_{H^m}\big)\cdot\big\|F\big\|_{H^m}.$$

Next, we have that:

$$\big\|[\nabla_4, P_k]F\big\|_{L^2} \leq \int_0^\infty |m_k|\cdot\big\|[\nabla_4, U(z)]F\big\|_{L^2}dz \leq \big\|F\big\|_{L^2}.$$

Similarly, we also have that:

$$\big\|\nabla[\nabla_4, P_k]F\big\|_{L^2} \leq \int_0^\infty |m_k|\cdot\big\|\nabla[\nabla_4, U(z)]F\big\|_{L^2}dz \leq \big\|F\big\|_{H^1}.$$

The proofs of (6.19) and (6.20) follow from (6.7) and (6.8). □

One disadvantage of the estimates (6.17)-(6.18) is that the right hand side cannot be summed in k, whereas the left hand side can be summed because of the presence of the P_k operator. We address this by proving a refined version of these estimates, which contains a different projection operator on the right hand side. The presence of different projection operators poses additional difficulties in the analysis of the model systems. We notice that this issue is a consequence of the nontrivial time dependence of the metrics $\not{g}$.

We define the symbol $\widetilde{m} \in \mathcal{M}_{\mathrm{op}}$ given by $\widetilde{m}(z) = zm(z)$, and we denote by $\widetilde{P}_k$ the associated projection operator. Moreover, we also introduce the projection operator $\underline{\widetilde{P}}_k$ which satisfies $\underline{\widetilde{P}}_k^2 = \widetilde{P}_k$.

Lemma 6.4. *We have the following estimates for $k \geq 0$:*

$$[\nabla_4, P_k]F = 2^{-2k}\chi\nabla^2\widetilde{P}_kF + O\Big(2^{-k}\big\|F\big\|_{L^2}\Big) \tag{6.21}$$

$$\big\|[\nabla_4, P_k]F\big\|_{L^2} \lesssim \big\|\underline{\widetilde{P}}_kF\big\|_{L^2} + 2^{-k}\big\|F\big\|_{L^2} \tag{6.22}$$

$$\nabla[\nabla_4, P_k]F = 2^{-2k}\chi\nabla^2\widetilde{P}_k\nabla F + O\Big(2^{-k}\big\|F\big\|_{H^1}\Big) \tag{6.23}$$

$$\big\|\nabla[\nabla_4, P_k]F\big\|_{L^2} \lesssim \big\|\underline{\widetilde{P}}_k\nabla F\big\|_{L^2} + 2^{-k}\big\|F\big\|_{H^1}. \tag{6.24}$$

Proof. In order to prove (6.21), we compute the following:

$$\begin{aligned}
[\nabla_4, P_k]F = \int_0^\infty m_k(z)[\nabla_4, U(z)]F dz = \int_0^\infty m_k(z)\int_0^z U(z-z')[\nabla_4, \Delta]U(z')F dz'dz \\
= \int_0^\infty m_k(z)\int_0^z U(z-z')\nabla\big(\chi\nabla U(z')F\big) + \\
+ \int_0^\infty m_k(z)\int_0^z U(z-z')\Big([\nabla_4, \Delta]U(z')F - \nabla\big(\chi\nabla U(z')F\big)\Big).
\end{aligned}$$

We can bound the second term in L^2 by:

$$\int_0^\infty |m_k(z)|\int_0^z \big\|U(z')F\big\|_{H^1}dz'dz \lesssim \big\|F\big\|_{L^2}\int_0^\infty |m_k(z)|\int_0^z (z')^{-1/2}dz'dz \lesssim 2^{-k}\big\|F\big\|_{L^2}.$$

Next, we write the first term as:

$$\int_0^\infty m_k(z)\int_0^z \nabla U(z-z')\big(\chi\nabla U(z')F\big) + \int_0^\infty m_k(z)\int_0^z [U(z-z'), \nabla]\big(\chi\nabla U(z')F\big). \tag{6.25}$$

The second term in (6.25) can be bounded in L^2 by:

$$\begin{aligned}
\int_0^\infty |m_k(z)|\int_0^z (z-z')^{1/2}\big\|\chi\nabla U(z')F\big\|_{L^2}dz'dz \\
\lesssim \big\|F\big\|_{L^2}\int_0^\infty |m_k(z)|\int_0^z (z-z')^{1/2}(z')^{-1/2}dz'dz \lesssim 2^{-k}\big\|F\big\|_{L^2}.
\end{aligned}$$

We write the first term in (6.25) as:

$$\int_0^\infty m_k(z)\int_0^z \nabla\big(\chi\cdot U(z-z')\nabla U(z')F\big) + \int_0^\infty m_k(z)\int_0^z \nabla[U(z-z'), \chi]\big(\nabla U(z')F\big).$$

The latter term in the above is bounded in L^2 by:

$$\int_0^\infty |m_k(z)| \int_0^z \big\|\nabla U(z')F\big\|_{L^2} dz'dz \lesssim \big\|F\big\|_{L^2} \int_0^\infty |m_k(z)| \int_0^z (z')^{-1/2} dz'dz \lesssim 2^{-k}\big\|F\big\|_{L^2}.$$

As a result, we proved that:

$$[\nabla_4, P_k]F = \chi \cdot \int_0^\infty m_k(z) \int_0^z \nabla U(z-z')\nabla U(z')F dz'dz + O\Big(2^{-k}\big\|F\big\|_{L^2}\Big)$$

$$= \chi \cdot \nabla^2 \int_0^\infty z m_k(z) U(z) F dz + \chi \cdot \int_0^\infty m_k(z) \int_0^z \nabla[U(z-z'), \nabla]U(z')F dz'dz + O\Big(2^{-k}\big\|F\big\|_{L^2}\Big)$$

$$= 2^{-2k}\chi \cdot \nabla^2 \int_0^\infty \widetilde{m}_k(z) U(z) F dz + O\Big(2^{-k}\big\|F\big\|_{L^2}\Big) = 2^{-2k}\chi\nabla^2 \widetilde{P}_k F + O\Big(2^{-k}\big\|F\big\|_{L^2}\Big).$$

The proof of (6.22) follows from (6.21), since $\underline{\widetilde{P}}_k^2 = \widetilde{P}_k$.

We now prove (6.23). We notice that using (6.13) we can write:

$$\nabla[\nabla_4, P_k]F = \int_0^\infty m_k(z) \int_0^z \nabla U(z-z')[\nabla_4, \Delta]U(z')F dz'dz$$

$$= \int_0^\infty m_k(z) \int_0^z \nabla U(z-z')\big(\chi\nabla^2 U(z')F\big) +$$

$$+ \int_0^\infty m_k(z) \int_0^z \nabla U(z-z')\Big([\nabla_4, \Delta]U(z')F - \chi\nabla^2 U(z')F\Big).$$

The second term can be bounded in L^2 by:

$$\int_0^\infty |m_k(z)| \int_0^z (z-z')^{-1/2} \big\|U(z')F\big\|_{H^1} dz'dz \lesssim 2^{-k}\big\|F\big\|_{H^1}.$$

The first term in the above expression can be written as:

$$\chi \int_0^\infty m_k(z) \int_0^z \nabla U(z-z')\nabla^2 U(z')F dz'dz + \nabla\chi \cdot \int_0^\infty m_k(z) \int_0^z U(z-z')\nabla^2 U(z')F dz'dz +$$

$$+ \int_0^\infty m_k(z) \int_0^z \nabla[U(z-z'), \chi]\nabla^2 U(z')F dz'dz.$$

To complete the proof of (6.23), we need to write this expression as $2^{-2k}\chi\nabla^2\widetilde{P}_k\nabla F + O\big(2^{-k}\|F\|_{H^1}\big)$. Indeed, we can bound the last two terms in L^2 by:

$$\int_0^\infty |m_k(z)| \int_0^z \big\|\nabla^2 U(z')F\big\|_{L^2} dz'dz \lesssim$$

$$\lesssim \int_0^\infty |m_k(z)| \int_0^z \big\|\nabla[\nabla, U(z')]F\big\|_{L^2} + \int_0^\infty |m_k(z)| \int_0^z \big\|\nabla U(z')\nabla F\big\|_{L^2}$$

$$\lesssim 2^{-2k}\big\|F\big\|_{L^2} + 2^{-k}\big\|\nabla F\big\|_{L^2}.$$

Finally, the leading term can be written as:

$$\chi\int_0^\infty m_k(z)\int_0^z \nabla U(z-z')\nabla U(z')\nabla F+\chi\int_0^\infty m_k(z)\int_0^z \nabla U(z-z')\nabla[\nabla,U(z')]F=$$

$$=\chi\int_0^\infty m_k(z)\int_0^z \nabla U(z-z')\nabla U(z')\nabla F+O\Big(2^{-k}\big\|F\big\|_{L^2}\Big)=2^{-2k}\chi\nabla^2\widetilde{P}_k\nabla F+O\Big(2^{-k}\big\|F\big\|_{H^1}\Big),$$

where in the last step we used the same argument as in the proof of (6.21). This completes the proof of (6.23). Moreover, the proof of (6.24) follows from (6.23) as before. □

We also prove estimates where we trade 1/2 derivatives on F for $2^{k/2}$ growth, which simplify certain error terms in the analysis of the top order singular component of the first model system in the low frequency regime:

Lemma 6.5. *We have the following estimates for* $k\geq 0$*:*

$$\big\|[\nabla_4,P_k]\nabla F\big\|_{L^2}\lesssim 2^{k/2}\big\|F\big\|_{H^{1/2}} \tag{6.26}$$

$$\big\|\nabla[\nabla_4,P_k]F\big\|_{L^2}\lesssim 2^{k/2}\big\|F\big\|_{H^{1/2}}. \tag{6.27}$$

Proof. In the previous proof we obtained the identity:

$$[\nabla_4,P_k]\nabla F=\int_0^\infty m_k(z)\int_0^z U(z-z')\nabla\big(\chi\nabla U(z')\nabla F\big)+$$

$$+\int_0^\infty m_k(z)\int_0^z U(z-z')\Big([\nabla_4,\Delta]U(z')\nabla F-\nabla\big(\chi\nabla U(z')\nabla F\big)\Big).$$

As a result, we get from Lemma 6.1:

$$\big\|[\nabla_4,P_k]\nabla F\big\|_{L^2}\lesssim\big\|F\big\|_{L^2}+\int_0^\infty|m_k(z)|\int_0^z\big((z-z')^{-1/2}+1\big)\big\|\nabla U(z')\nabla F\big\|_{L^2}dz'dz$$

$$\lesssim\big\|F\big\|_{L^2}+\int_0^\infty|m_k(z)|\int_0^z\big((z-z')^{-1/2}+1\big)(z')^{-3/4}\big\|F\big\|_{H^{1/2}}dz'dz$$

$$\lesssim\big\|F\big\|_{H^{1/2}}\Big(1+\int_0^\infty|m_k(z)|z^{-1/4}dz\Big).$$

A similar proof also gives:

$$\big\|\nabla[\nabla_4,P_k]F\big\|_{L^2}\lesssim\big\|F\big\|_{L^2}+\int_0^\infty|m_k(z)|\int_0^z(z-z')^{-1/2}\big\|\nabla^2U(z')F\big\|_{L^2}dz'dz\lesssim 2^{k/2}\big\|F\big\|_{H^{1/2}}.$$

□

Convention. For the remainder of the thesis, we fix the projection operator P_k to satisfy $\sum_k P_k^2 = I$. We notice that all the estimates established above in this section are valid for any LP projections with symbols in $\mathcal{M}_{\text{op}}$.

In order to control a top order bulk term with bad sign in the high frequency estimate for the second model system, we need a refined Poincaré inequality for LP projections. The key aspect of this result is that the projection operators on the right hand side have the same symbol as the one on the left hand side. Moreover, all the frequencies higher than k are contained in the last term, which is lower order.

Lemma 6.6. *For any $k \geq 0$, and $\delta > 0$, we have the inequality:*

$$\|P_kF\|_{L^2}^2 \lesssim \frac{1}{\delta}2^{-2k}\|\nabla P_kF\|_{L^2}^2 + \delta \sum_{0\leq l<k} 2^{-9k+7l}\|\nabla P_lF\|_{L^2}^2 + \delta^{-1}2^{-4k}\|F\|_{L^2}^2. \tag{6.28}$$

Proof. Let $\dot{P}$ be the projection operator with symbol given by $\dot{m}(z) = -\int_z^\infty m(z')dz'$. According to the proof of [KR06, Theorem 5.5 (v)] we have that $2^{2k}P_kF = \Delta\dot{P}_kF$. This implies the Poincaré inequality:

$$\|P_kF\|_{L^2} \lesssim \sqrt{\delta}2^{-k}\|\nabla\dot{P}_kF\|_{L^2} + \frac{1}{\sqrt{\delta}}2^{-k}\|\nabla P_kF\|_{L^2}.$$

Also, for all $l \geq k \geq 0$ we have the following estimate, according to the proof of [KR06, Theorem 5.5 (ii)]:

$$\|P_l\dot{P}_kF\|_{L^2} \lesssim 2^{-2(l-k)}\|P_kF\|_{L^2}. \tag{6.29}$$

We use these two bounds, together with the other usual bounds for LP projections, to get:

$$\begin{aligned}\|P_kF\|_{L^2} &\lesssim \sqrt{\delta}2^{-k}\|\nabla\dot{P}_kF\|_{L^2} + \frac{1}{\sqrt{\delta}}2^{-k}\|\nabla P_kF\|_{L^2}\\ &\lesssim \sqrt{\delta}2^{-k}\|\dot{P}_k\nabla F\|_{L^2} + \frac{1}{\sqrt{\delta}}2^{-2k}\|F\|_{L^2} + \frac{1}{\sqrt{\delta}}2^{-k}\|\nabla P_kF\|_{L^2}\\ &\lesssim \sqrt{\delta}\sum_{l\in\mathbb{Z}} 2^{-k}\|P_l^2\dot{P}_k\nabla F\|_{L^2} + \frac{1}{\sqrt{\delta}}2^{-2k}\|F\|_{L^2} + \frac{1}{\sqrt{\delta}}2^{-k}\|\nabla P_kF\|_{L^2}.\end{aligned}$$

For the first term on the RHS we have by the L^2-almost orthogonality, (6.29), and (6.15):

$$\begin{aligned}&\sum_{l\in\mathbb{Z}} 2^{-k}\|P_l^2\dot{P}_k\nabla F\|_{L^2} \lesssim\\ &\lesssim \sum_{0\leq l<k} 2^{-k-4|k-l|}\|P_l\nabla F\|_{L^2} + 2^{-k}\|P_k\nabla F\|_{L^2} + \sum_{l>k} 2^{-k-2(l-k)}\|P_k\nabla F\|_{L^2}\end{aligned}$$

$$\lesssim 2^{-k}\|\nabla P_k F\|_{L^2}+2^{-2k}\|F\|_{L^2}+\sum_{0\leq l<k} 2^{-k-4|k-l|}\|P_l\nabla F\|_{L^2}.$$

As a result, we obtain that:

$$\|P_k F\|_{L^2}\lesssim \frac{1}{\sqrt{\delta}}2^{-k}\|\nabla P_k F\|_{L^2}+\sqrt{\delta}\sum_{0\leq l<k} 2^{-|k-l|/2}\cdot 2^{-k-7|k-l|/2}\|P_l\nabla F\|_{L^2}+\frac{1}{\sqrt{\delta}}2^{-2k}\|F\|_{L^2}.$$

We square this inequality and use Cauchy-Schwarz in order to conclude. □

We introduce the $(\log\nabla)$ operator, essential for the renormalization of h to $\mathfrak{h}$. We define for any smooth tensor F on S_0 and $k\geq 0$:

$$(\log\nabla)F=\sum_{l\geq 0} P_l^2 F\cdot \log 2^l.$$

We prove the following bound on the $(\log\nabla)$ operator, used in lower order estimates:

Lemma 6.7. *Set*[2] $\eta=1/10$. *For any smooth horizontal tensor* F, *and any* $s\geq 0$, *we have the estimate:*

$$\|(\log\nabla)F\|_{H^s}\lesssim\|F\|_{H^{s+\eta}}.$$

Proof. We first notice that we have the inequality:

$$\|(\log\nabla)F\|_{L^2}^2\lesssim\left(\sum_{k\geq 0}\|P_k^2F\cdot\log 2^k\|_{L^2}\right)^2$$

$$\lesssim\left(\sum_{k\geq 0}2^{k\eta/2}\|P_kF\|_{L^2}\right)^2\lesssim\sum_{k\geq 0}2^{2k\eta}\|P_kF\|_{L^2}^2\lesssim\|F\|_{H^\eta}.$$

Using this, we can also bound the following:

$$\|(\log\nabla)F\|_{H^s}^2\lesssim\|(\log\nabla)F\|_{L^2}^2+\sum_{l\geq 0}2^{2sl}\|P_l(\log\nabla)F\|_{L^2}^2$$

$$\lesssim\|(\log\nabla)F\|_{L^2}^2+\sum_{l\geq 0}2^{2sl}\|(\log\nabla)P_lF\|_{L^2}^2\lesssim\|F\|_{H^\eta}+\sum_{l,k\geq 0}2^{2sl}2^{2\eta k}\|P_kP_lF\|_{L^2}^2$$

$$\lesssim\|F\|_{H^\eta}+\sum_{k\geq 0}\sum_{l=0}^{k}2^{2(s+\eta)k}\|P_lP_kF\|_{L^2}^2+\sum_{l\geq 0}\sum_{k=0}^{l}2^{2(s+\eta)l}\|P_kP_lF\|_{L^2}^2$$

$$\lesssim\|F\|_{H^\eta}+\sum_{k\geq 0}2^{2(s+\eta)k}\|P_kF\|_{L^2}^2\lesssim\|F\|_{H^{s+\eta}}.$$

□

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[2] This result holds for any $\eta>0$, with implicit constant depending on η. For our purposes it suffices to fix $\eta=1/10$.

Next, we define the following operator for $k \geq 0$:

$$R_k F = 2P_k(\log \nabla)F - 2\log 2^k \cdot P_k F = 2\sum_{l\geq 0} \log 2 \cdot (l-k) \cdot P_k P_l^2 F - 2\sum_{l<0} \log 2^k \cdot P_k P_l^2 F. \quad (6.30)$$

The operator R_k appears as a commutation error term when projecting the expansion at $\{\tau = 0\}$ of the singular component of Φ_0 in the analysis of the first model system.

We consider the projection operator $\underline{P}_k$ which satisfies $\underline{P}_k^2 = P_k$. We have the estimates for R_k :

Lemma 6.8. *Let F be any smooth tensor on S_0. We extend $R_k F$ to $(0,1] \times S^n$ to be independent of τ. We also denote $t = 2^k \tau$. Then, for any $k \geq 0$ we have:*

$$\big\|\Delta_{\not{g}_\tau} R_k F\big\|_{L^2(S_\tau)} \lesssim 2^k \big\|\underline{P}_k F\big\|_{H^1(S_0)} \quad (6.31)$$

$$\big\|\nabla R_k F\big\|_{L^2(S_\tau)} \lesssim \big\|\underline{P}_k F\big\|_{H^1(S_0)} \quad (6.32)$$

$$2^k \big\|R_k F\big\|_{L^2(S_\tau)} \lesssim \big\|\underline{P}_k F\big\|_{H^1(S_0)} \quad (6.33)$$

$$\big\|\nabla_t R_k F\big\|_{L^2(S_\tau)} \lesssim 2^{-3k} t \big\|\underline{P}_k F\big\|_{H^1(S_0)} \quad (6.34)$$

$$\big\|\nabla\nabla_t R_k F\big\|_{L^2(S_\tau)} \lesssim 2^{-2k} t \big\|\underline{P}_k F\big\|_{H^1(S_0)}. \quad (6.35)$$

Proof. For the purpose of this proof we denote by $\Delta_{\not{g}_\tau}$ and $\Delta_{\not{g}_0}$ the Laplace-Beltrami operators on $\big(S_\tau, \not{g}_\tau\big)$ and $\big(S_0, \not{g}_0\big)$. Because of (5.15), we can express the LHS of (6.31) using derivatives at $\{\tau = 0\}$:

$$2^{-k}\big\|\Delta_{\not{g}_\tau} R_k F\big\|_{L^2(S_\tau)} \lesssim 2^{-k}\big\|R_k F\big\|_{H^2(S_\tau)}$$

$$\lesssim 2^{-k}\big\|R_k F\big\|_{H^2(S_0)} \lesssim 2^{-k}\big\|\Delta_{\not{g}_0} R_k F\big\|_{L^2(S_0)} + 2^{-k}\big\|R_k F\big\|_{H^1(S_0)}.$$

Using (5.15) and the finite band property of [KR06] we have:

$$2^k\big\|R_k F\big\|_{L^2(S_\tau)} \lesssim 2^k \big\|R_k F\big\|_{L^2(S_0)}$$

$$\lesssim \sum_{l\geq 0} 2^k |l-k| \cdot \big\|\underline{P}_k P_l P_l \underline{P}_k F\big\|_{L^2(S_0)} + \sum_{l<0} k 2^k \cdot \big\|\underline{P}_k P_l P_l \underline{P}_k F\big\|_{L^2(S_0)} \lesssim$$

$$\lesssim \sum_{l\geq 0} \frac{2^{k-l}|l-k|}{2^{2|l-k|}} \cdot \big\|\nabla \underline{P}_k F\big\|_{L^2(S_0)} + \sum_{l<0} k 2^{-k+2l} \cdot \big\|\underline{P}_k F\big\|_{L^2(S_0)} \lesssim \big\|\underline{P}_k F\big\|_{H^1(S_0)},$$

$$2^{-k}\big\|\Delta_{\not{g}_0} R_k F\big\|_{L^2(S_0)} \lesssim \sum_{l\geq 0} 2^{-k}|l-k| \cdot \big\|\underline{P}_k P_l \Delta_{\not{g}_0} P_l \underline{P}_k F\big\|_{L^2(S_0)} + \sum_{l<0} k 2^{-k} \cdot \big\|\underline{P}_k P_l \Delta_{\not{g}_0} P_l \underline{P}_k F\big\|_{L^2}$$

$$\lesssim \sum_{l\geq 0} \frac{2^{l-k}|l-k|}{2^{2|l-k|}} \cdot \big\|\nabla \underline{P}_k F\big\|_{L^2(S_0)} + \sum_{l<0} k2^{-3k+l} \cdot \big\|\nabla \underline{P}_k F\big\|_{L^2(S_0)} \lesssim \big\|\underline{P}_k F\big\|_{H^1(S_0)}.$$

Combining these two estimates (again using (5.15) as well), we also have that:

$$\big\|\nabla R_k F\big\|_{L^2(S_\tau)} \lesssim \big\|R_k F\big\|_{H^1(S_0)} \lesssim \big\|\underline{P}_k F\big\|_{H^1(S_0)}.$$

So far we proved (6.31), (6.32), and (6.33). In order to prove (6.34) and (6.35), we recall formula (5.12). Using this and the fact that $R_k F$ is independent of τ, we get:

$$\big\|\nabla_t R_k F\big\|_{L^2(S_\tau)} \lesssim \big\|(\nabla_t - \mathcal{L}_t) R_k F\big\|_{L^2(S_\tau)} \lesssim \frac{t}{2^{2k}} \big\|R_k F\big\|_{L^2(S_\tau)} \lesssim \frac{t}{2^{3k}} \big\|\underline{P}_k F\big\|_{H^1(S_0)}.$$

Finally, again using that $R_k F$ is independent of τ, we have:

$$\big\|\nabla\nabla_t R_k F\big\|_{L^2(S_\tau)} \lesssim \big\|\nabla(\nabla_t - \mathcal{L}_t) R_k F\big\|_{L^2(S_\tau)} \lesssim \frac{t}{2^{2k}} \big\|R_k F\big\|_{H^1(S_\tau)} \lesssim \frac{t}{2^{2k}} \big\|\underline{P}_k F\big\|_{H^1(S_0)}.$$

□

Finally, we prove the following result which implies that it is equivalent whether we project the expansions at $\tau = 0$ with respect to $\not{g}_0 = \not{g}(0)$ or $\not{g}_\tau = \not{g}(\tau)$.

Lemma 6.9. *We consider F to be a smooth tensor on S_0, extended to be independent of τ. Denote by ${}^{(\not{g}_0)}P_k$ the projection with respect to $\big(S_0^n, \not{g}_0\big)$ and by ${}^{(\not{g}_\tau)}P_k$ the projection with respect to $\big(S_\tau^n, \not{g}_\tau\big)$. Then we have the estimate for any $s \geq 0$, $k \geq 0$:*

$$\big\|\big({}^{(\not{g}_0)}P_k - {}^{(\not{g}_\tau)}P_k\big)F\big\|_{H^s} \lesssim_s \tau^2 \big\|F\big\|_{H^{s+2}}. \tag{6.36}$$

Proof. The bound follows using Duhamel's formula as in [KR05]:

$$\begin{aligned}\big\|\big({}^{(\not{g}_0)}P_k - {}^{(\not{g}_\tau)}P_k\big)F\big\|_{H^s} &\lesssim \int_0^\infty |m_k| \cdot \big\|{}^{(\not{g}_0)}U(z)F - {}^{(\not{g}_\tau)}U(z)F\big\|_{H^s} dz \\ &\lesssim \int_0^\infty |m_k| \int_0^z \Big\|{}^{(\not{g}_0)}U(z-z')\big(\Delta_{\not{g}_\tau} - \Delta_{\not{g}_0}\big)\,{}^{(\not{g}_\tau)}U(z')F\Big\|_{H^s} dz' dz \\ &\lesssim_s \int_0^\infty |m_k| \int_0^z \tau^2 \big\|{}^{(\not{g}_\tau)}U(z')F\big\|_{H^{s+2}} dz' dz,\end{aligned}$$

where we also used the expansion of $\not{g}_\tau$ at $\tau = 0$. The last term is bounded by $\tau^2 \big\|F\big\|_{H^{s+2}}$. □

7 Estimates for the First Model System

One of the central parts of our argument is proving estimates for the first model system (5.13), in terms of the asymptotic data at $\{\tau = 0\}$. This system includes the commuted Bianchi system

(5.11), where Φ_0 and Φ_i correspond to the commuted curvature components $\nabla_4^{\frac{n-4}{2}}\alpha$ and $\nabla_4^{\frac{n-4}{2}}\Psi^G$. The analysis of the first model system (5.13) implies estimates on the solution $\big(\mathcal{M},g\big)$ at finite self-similar times in terms of the data at $\{v=0\}$.

In the present section, we prove in Theorem 7.1 the main estimates for the system (5.13). This section is based on [Cic24, Section 7] and [Cic26, Section 3]. We encourage the reader to return to Section 1.3.3 of the introduction for an outline of the proof.

Theorem 7.1. *Let $M>N$ be large enough. We assume that $\Phi_0,\ldots,\Phi_I$ satisfy the first model system on the hypersurface $\{u=-1\}\times\{\tau\in(0,1)\}\times S^n$ of the spacetime $\big(\mathcal{M},g\big)$ obtained in Theorem 3.1.*

For all $\tau\in(0,1]$, we define the energy of the solution on S_τ to be:

$$\begin{aligned}\mathcal{E}_I(\tau)=&\tau^2\big\|\nabla_\tau\nabla^M\Phi_0\big\|^2_{H^{1/2}(S_\tau)}+\tau^2\big\|\nabla^M\Phi_0\big\|^2_{H^{3/2}(S_\tau)}+\\&+\sum_{i=1}^{I}\Big(\tau\big\|\nabla_\tau\nabla^M\Phi_i\big\|^2_{H^{1/2}(S_\tau)}+\tau\big\|\nabla^{M+1}\Phi_i\big\|^2_{H^{1/2}(S_\tau)}+\big\|\Phi_i\big\|^2_{H^{M+1}(S_\tau)}\Big).\end{aligned}$$

We define the asymptotic data norm and the inhomogeneous norm as:

$$\mathcal{D}_I=\big\|\mathcal{O}\big\|^2_{H^{M+1}(S^n)}+\big\|\mathfrak{h}\big\|^2_{H^{M+1}(S^n)}+\sum_{i=1}^{I}\big\|\Phi_i^0\big\|^2_{H^{M+1}(S^n)},$$

$$\mathcal{F}_I(\tau)=\sum_{m=0}^{M}\sum_{i=0}^{I}\int_0^\tau\big\|F^i_m\big\|^2_{L^2(S_{\tau'})}d\tau'+\sum_{i=0}^{I}\int_0^\tau\tau'\big\|F^i_M\big\|^2_{H^{1/2}(S_{\tau'})}d\tau'.$$

The solution of the first model system satisfies the estimates for $\tau\in(0,1]$ and a constant $C_I>0$:

$$\mathcal{E}_I(\tau)\leq C_I\mathcal{D}_I+C_I\mathcal{F}_I(\tau),\tag{7.1}$$

$$\big\|\Phi_0\big\|^2_{H^{M+1}(S_\tau)}\leq C_I\mathcal{D}_I+C_I\mathcal{F}_I(\tau)+C_I|\log\tau|^2\big\|\mathcal{O}\big\|^2_{H^{M+1}(S^n)},\tag{7.2}$$

where the constant C_I depends on $M>N$, the bounds on $\big\|\slashed{Riem}(\slashed{g}_0)\big\|_{H^M}$, and the bounds satisfied by the background $\big(\mathcal{M},g\big)$ according to Theorem 3.1.

In order to prove this result, we follow a similar strategy to [Cic23] and we decompose Φ_0 into its singular and regular components. We have for each $m\leq M$:

$$\nabla^m\Phi_0=\big(\nabla^m\Phi_0\big)_Y+\big(\nabla^m\Phi_0\big)_J,\tag{7.3}$$

where we define the singular component $\big(\nabla^m\Phi_0\big)_Y$ to be the horizontal tensor that solves the linear equation:

$$\nabla_\tau\big(\nabla_\tau\big(\nabla^m\Phi_0\big)_Y\big)+\frac{1}{\tau}\nabla_\tau\big(\nabla^m\Phi_0\big)_Y-4\Delta\big(\nabla^m\Phi_0\big)_Y=\psi\nabla\big(\nabla^m\Phi_0\big)_Y \tag{7.4}$$

$$\big(\nabla^m\Phi_0\big)_Y(\tau)=2\nabla^m\mathcal{O}\log(\tau)+2(\log\nabla)\nabla^m\mathcal{O}+O\big(\tau^2|\log(\tau)|^2\big),$$

$$\nabla_\tau\big(\nabla^m\Phi_0\big)_Y(\tau)=\frac{2\nabla^m\mathcal{O}}{\tau}+O\big(\tau|\log(\tau)|^2\big),$$

and we also define as above $(\log\nabla)\nabla^m\mathcal{O}=\sum_{k\geq0}P_k^2\nabla^m\mathcal{O}\cdot\log 2^k,\ \mathfrak{h}_m=\nabla^m h-2(\log\nabla)\nabla^m\mathcal{O}$. We define the regular component of $\nabla^m\Phi_0$ by $\big(\nabla^m\Phi_0\big)_J=\nabla^m\Phi_0-\big(\nabla^m\Phi_0\big)_Y$. This satisfies the equations:

$$\nabla_\tau\big(\nabla_\tau\big(\nabla^m\Phi_0\big)_J\big)+\frac{1}{\tau}\nabla_\tau\big(\nabla^m\Phi_0\big)_J-4\Delta\big(\nabla^m\Phi_0\big)_J=\psi\nabla\big(\nabla^m\Phi_0\big)_J+\sum_{j=1}^{I}\psi\nabla\nabla^m\Phi_j+F_m^0 \tag{7.5}$$

$$\big(\nabla^m\Phi_0\big)_J(\tau)=\mathfrak{h}_m+O\big(\tau^2|\log(\tau)|^2\big),\ \nabla_\tau\big(\nabla^m\Phi_0\big)_J(\tau)=O\big(\tau|\log(\tau)|^2\big)\text{ in }C^\infty(S^n). \tag{7.6}$$

The notation for the regular and singular components is based on the similarities to the first and second Bessel functions $J_0,\ Y_0$, as in the case of [Cic23].

Since the singular component of Φ_0 decouples from the rest of the system, an essential step in our argument consists of establishing the following top order estimates:

Theorem 7.2. *For any $M>0$ large enough, the singular component satisfies the estimates for all $\tau\in(0,1]$:*

$$\tau^2\big\|\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|^2_{H^{1/2}}+\tau^2\big\|\nabla\big(\nabla^M\Phi_0\big)_Y\big\|^2_{H^{1/2}}\lesssim\big\|\mathcal{O}\big\|^2_{H^{M+1}}, \tag{7.7}$$

$$\sum_{m=0}^{M}\big\|\big(\nabla^m\Phi_0\big)_Y\big\|^2_{H^1}\lesssim\big(1+|\log\tau|^2\big)\big\|\mathcal{O}\big\|^2_{H^{M+1}}. \tag{7.8}$$

We outline the structure of the rest of the section. In Section 7.1 we prove in Proposition 7.1 a general result for the existence of solutions of (7.9) with asymptotic data at $\mathcal{I}^-$, which in particular implies the decomposition (7.3). In Section 7.2 we prove lower order estimates. We prove in Section 7.2.1 in Propositions 7.2 and 7.3 the lower order estimates (1.39) and (1.40). We also prove estimates for the commutator term $\mathcal{C}=\big(\nabla^M\Phi_0\big)_Y-\nabla\big(\nabla^{M-1}\Phi_0\big)_Y$ and $H^{1/2}$ estimates for $\nabla(\nabla^{M-1}\Phi_0)_Y$ and $\nabla_\tau(\nabla^{M-1}\Phi_0)_Y$ in Section 7.2.2. The goal of Section 7.3 is to prove Theorem 7.2 for the singular component. At top order, we consider each P_k projection of the solution, and we treat separately the low frequency regime in Section 7.3.1 and the high frequency regime in Section 7.3.2. In Section 7.3.3 we combine the low frequency regime and the high frequency regime estimates to prove Theorem 7.2. In Section 7.4 we prove the top order estimates for the regular quantities. We prove the low frequency regime estimates for the regular quantities in Section 7.4.1, and the corresponding high frequency regime estimates in Section 7.4.2. Finally, we combine the estimates in Section 7.4.3 to complete the proof of Theorem 7.1.

7.1 Construction of the Singular Component

In this section, we prove an existence and uniqueness result for solutions of (7.9) with asymptotic data at $\mathcal{I}^-$. In particular, this implies the existence and uniqueness of the singular component defined by (7.4).

We first remark that we frequently use ∇_τ as a multiplier to obtain energy estimates. The following lemma implies that the additional terms resulting from differentiating the volume form or the metric can be controlled using Grönwall for $\tau \in (0, 1]$. We point out that we usually bound these terms implicitly.

Lemma 7.1. *For any smooth horizontal tensor F defined on $\mathcal{M}$, we have:*

$$\frac{1}{2}\frac{d}{d\tau}\|F\|^2_{L^2(S_\tau)} = \int_{S_\tau} F \cdot \nabla_\tau F dVol_{\slashed{g}} + O\Big(\tau\|F\|^2_{L^2(S_\tau)}\Big).$$

Proof. We denote $v = \sqrt{\tau}$, and compute that $\partial_\tau = 2\tau\partial_v$ and $e_4 = \partial_v$. Since $\chi = \mathcal{L}_{e_4}\slashed{g}$, we use the standard formula for $\frac{d}{dv}\int_{S^n}|F|^2(v)dVol_{\slashed{g}}$ and the bounds on the background in (5.16) to get:

$$\frac{1}{2}\frac{d}{d\tau}\int_{S_\tau}|F|^2 = \frac{1}{2}\int_{S_\tau}\nabla_\tau|F|^2 + \int_{S_\tau}\tau\mathrm{tr}\chi|F|^2 = \int_{S_\tau} F \cdot \nabla_\tau F + O\Big(\tau\|F\|^2_{L^2(S_\tau)}\Big).$$

□

The following result implies that we can decompose the solution $\nabla^m\Phi_0$ into its regular and singular components, for each $m \leq M$.

Proposition 7.1. *For any $\Phi^0, \Phi^1 \in C^\infty(S^n)$ smooth tensors, there exists a unique solution on $\mathcal{M}$ of:*

$$\nabla_\tau\big(\nabla_\tau\Phi\big) + \frac{1}{\tau}\nabla_\tau\Phi - 4\Delta\Phi = \psi\nabla\Phi, \tag{7.9}$$

satisfying the following asymptotic expansion as $\tau \to 0$:

$$\Phi(\tau) = \Phi^0\log(\tau) + \Phi^1 + O\big(\tau^2|\log(\tau)|^2\big),\ \nabla_\tau\Phi(\tau) = \frac{\Phi^0}{\tau} + O\big(\tau|\log(\tau)|^2\big)\ \textit{in}\ C^\infty(S^n).$$

Proof. It suffices to prove that for any $K > 0$ there exists a unique solution of (7.9) such that the above expansions hold in $H^K(S^n)$. We introduce the quantity $\widetilde{\Phi} = \Phi - \Phi^0 \log(\tau) - \Phi^1$, which satisfies the equation:

$$\nabla_\tau\big(\nabla_\tau\widetilde{\Phi}\big) + \frac{1}{\tau}\nabla_\tau\widetilde{\Phi} - 4\Delta\widetilde{\Phi} = \psi\nabla\widetilde{\Phi} + F_1(\Phi^0, \Phi^1)\cdot\log(\tau) + F_2(\Phi^0, \Phi^1). \tag{7.10}$$

where F_1 and F_2 are bounded functions of Φ^0, Φ^1, and their angular derivatives. To obtain this equation we use the fact that Φ^0 and Φ^1 are Lie transported in time, so by (5.12) we have $\nabla_\tau\Phi^0 = \tau\chi\cdot\Phi_0$ and $\nabla_\tau\Phi^1 = \tau\chi\cdot\Phi_1$. For any $k \geq 0$ we obtain the commuted equation:

$$\nabla_\tau\big(\nabla^k\nabla_\tau\widetilde{\Phi}\big) + \frac{1}{\tau}\nabla^k\nabla_\tau\widetilde{\Phi} - 4\Delta\nabla^k\widetilde{\Phi} = \psi\nabla\nabla^k\widetilde{\Phi} + F_1^k(\Phi^0, \Phi^1)\cdot\log(\tau) + F_2^k(\Phi^0, \Phi^1) + F_3^k(\widetilde{\Phi}),$$

where we denote the error terms:

$$F_1^k(\Phi^0, \Phi^1) = \nabla^k\big(F_1(\Phi^0, \Phi^1)\big),\ F_2^k(\Phi^0, \Phi^1) = \nabla^k\big(F_2(\Phi^0, \Phi^1)\big),$$

$$F_3^k(\widetilde{\Phi}) = [\nabla_\tau, \nabla^k]\nabla_\tau\widetilde{\Phi} - 4[\Delta, \nabla^k]\widetilde{\Phi} - [\psi\nabla, \nabla^k]\widetilde{\Phi}.$$

We remark that using the smoothness of Φ^0, Φ^1, and the background metric $\slashed{g}$, we obtain that:

$$F_1^k(\Phi^0, \Phi^1)\cdot\log(\tau) + F_2^k(\Phi^0, \Phi^1) = O_k\big(1 + |\log\tau|\big).$$

We define $\widetilde{\Phi}_\epsilon$ to be the solution to (7.10) on $[\epsilon, 1]\times S^n$ with initial data $\widetilde{\Phi}_\epsilon|_{\tau=\epsilon} = \nabla_\tau\widetilde{\Phi}_\epsilon|_{\tau=\epsilon} = 0$. Contracting the commuted equation for $\widetilde{\Phi}_\epsilon$ with $\nabla^k\nabla_\tau\widetilde{\Phi}_\epsilon$, we obtain the standard energy estimate:

$$\big\|\nabla^k\nabla_\tau\widetilde{\Phi}_\epsilon\big\|_{L^2}^2 + \big\|\nabla\nabla^k\widetilde{\Phi}_\epsilon\big\|_{L^2}^2 \lesssim_k \int_\epsilon^\tau \big\|\nabla^k\nabla_\tau\widetilde{\Phi}_\epsilon\big\|_{L^2}\big\|\nabla\nabla^k\widetilde{\Phi}_\epsilon\big\|_{L^2} +$$

$$\int_\epsilon^\tau \big\|\nabla\nabla^k\widetilde{\Phi}_\epsilon\big\|_{L^2}\big\|[\nabla^{k+1}, \nabla_\tau]\widetilde{\Phi}_\epsilon\big\|_{L^2} + \int_\epsilon^\tau |\log(\tau')|\cdot\big\|\nabla^k\nabla_\tau\widetilde{\Phi}_\epsilon\big\|_{L^2}\big\|F_1^k(\Phi_0, \Phi_1)\big\|_{L^2}$$

$$+ \int_\epsilon^\tau \big\|\nabla^k\nabla_\tau\widetilde{\Phi}_\epsilon\big\|_{L^2}\big\|F_2^k(\Phi_0, \Phi_1)\big\|_{L^2} + \int_\epsilon^\tau \big\|\nabla^k\nabla_\tau\widetilde{\Phi}_\epsilon\big\|_{L^2}\big\|F_3^k(\widetilde{\Phi}_\epsilon)\big\|_{L^2}.$$

We notice that in the above estimate we dropped the bulk term with a favorable sign. Also, it is essential that $\widetilde{\Phi}_\epsilon$ vanishes to sufficiently high order at $\tau = 0$ in order to not have any initial data contribution. Moreover, due to Lemma 7.1, we also have the terms $\int_\epsilon^\tau \tau'\|\nabla\nabla^k\widetilde{\Phi}_\epsilon\|_{L^2}^2$ and $\int_\epsilon^\tau \tau'\|\nabla^k\nabla_\tau\widetilde{\Phi}_\epsilon\|_{L^2}^2$ on the RHS, but these can be bounded using Grönwall.

Using the higher order version of the commutation formula (6.12), the smoothness of the background metric $\slashed{g}$, and the Grönwall inequality, we obtain the estimate:

$$\big\|\nabla^k\nabla_\tau\widetilde{\Phi}_\epsilon\big\|_{L^2}^2 + \big\|\nabla\nabla^k\widetilde{\Phi}_\epsilon\big\|_{L^2}^2 \lesssim_k$$

$$\lesssim_k \int_\epsilon^\tau \big\|\widetilde{\Phi}_\epsilon\big\|^2_{H^{k+1}} d\tau' + \int_\epsilon^\tau \big\|\nabla_\tau\widetilde{\Phi}_\epsilon\big\|^2_{H^k} d\tau' + \int_\epsilon^\tau \big\|\nabla_\tau\widetilde{\Phi}_\epsilon\big\|_{H^k}\cdot\big(1+|\log\tau'|\big)d\tau'.$$

where the implicit constant depends on $k\geq 0$. On the other hand, using Lemma 7.1, Grönwall, and the commutation formulas, we also have:

$$\big\|\widetilde{\Phi}_\epsilon\big\|^2_{H^k} \lesssim_k \int_\epsilon^\tau \big\|\widetilde{\Phi}_\epsilon\big\|^2_{H^k} d\tau' + \int_\epsilon^\tau \big\|\nabla_\tau\widetilde{\Phi}_\epsilon\big\|^2_{H^k} d\tau'.$$

We use our previous two estimates and Grönwall to obtain that for any $\tau\in[\epsilon,1]$:

$$\big\|\nabla_\tau\widetilde{\Phi}_\epsilon\big\|^2_{H^k} + \big\|\widetilde{\Phi}_\epsilon\big\|^2_{H^{k+1}} \lesssim_k \int_\epsilon^\tau \big\|\nabla_\tau\widetilde{\Phi}_\epsilon\big\|_{H^k}\cdot\big(1+|\log\tau'|\big)d\tau'.$$

By taking the supremum on $[\epsilon,\tau]$ in the above inequality for each $\tau\in[\epsilon,1]$, we obtain that:

$$\big\|\nabla_\tau\widetilde{\Phi}_\epsilon\big\|_{H^k} = O_k\big(\tau(1+|\log\tau|)\big).$$

We use this and a similar argument for $\big\|\widetilde{\Phi}_\epsilon\big\|_{H^k}$ to obtain the bound:

$$\big\|\widetilde{\Phi}_\epsilon\big\|_{H^k} = O_k\big(\tau^2(1+|\log\tau|)^2\big).$$

Using the Banach-Alaoglu theorem and compactness, we obtain that for $K\ll k$ there exists $\widetilde{\Phi}$ a solution of (7.10) on $\mathcal{M}$ such that:

$$\big\|\nabla_\tau\widetilde{\Phi}\big\|_{H^K} = O_K\big(\tau(1+|\log\tau|)^2\big),\ \big\|\widetilde{\Phi}\big\|_{H^K} = O_K\big(\tau^2(1+|\log\tau|)^2\big).$$

As a result, we obtain that $\Phi=\widetilde{\Phi}+\Phi^0\log(\tau)+\Phi^1$ is a solution of (7.9) and satisfies the desired expansions. Finally, we remark that uniqueness follows by using our standard energy estimate on $[0,\tau]$ for the difference of two solutions with the same asymptotic expansion. □

7.2 Lower Order Estimates

The goal of this section is to establish estimates that are lower order in terms of the number of angular derivatives. We point out that the estimates of this section are not sharp, but it is essential that we use only the quantities that appear on the right hand side of the estimates in Theorem 7.1.

The lower order estimates are carried out in two parts. First, we prove estimates for an integer number of angular derivatives $m<M$, and the bounds for the commutator term $\mathcal{C}$. We then also prove estimates for $M-\frac{1}{2}$ derivatives of the singular component $\big(\Phi_0\big)_Y$.

7.2.1 Standard Estimates

We first prove the lower order estimates for the singular component using the strategy outlined in Section 1.3.3. We further decompose for every $m < M$:

$$\big(\nabla^m \Phi_0\big)_Y = \nabla^m \Phi^1_{0Y} + \nabla^m \Phi^2_{0Y},$$

where using Proposition 7.1 we have that $\nabla^m \Phi^1_{0Y},\ \nabla^m \Phi^2_{0Y}$ are the solutions of (7.4) such that:

$$\nabla^m \Phi^1_{0Y}(\tau) = 2\nabla^m \mathcal{O} \log(\tau) + O\big(\tau^2 |\log(\tau)|^2\big),\ \nabla_\tau \nabla^m \Phi^1_{0Y}(\tau) = \frac{2\nabla^m \mathcal{O}}{\tau} + O\big(\tau |\log(\tau)|^2\big),$$

$$\nabla^m \Phi^2_{0Y}(\tau) = 2(\log \nabla)\nabla^m \mathcal{O} + O\big(\tau^2 |\log(\tau)|^2\big),\ \nabla_\tau \nabla^m \Phi^2_{0Y}(\tau) = O\big(\tau |\log(\tau)|^2\big).$$

Using this decomposition, we prove the following lower order estimates on the singular component:

Proposition 7.2. *Set $\eta = 1/10$. The singular component satisfies the following estimates for any $m < M$:*

$$\left\| \nabla_\tau \frac{\nabla^m \Phi^1_{0Y}}{\log \tau} \right\|^2_{L^2} + \left\| \frac{\nabla^m \Phi^1_{0Y}}{\log \tau} \right\|^2_{H^1} \lesssim \big\|\mathcal{O}\big\|^2_{H^{m+1}} \textit{ for } \tau \in \left(0, \frac{1}{2}\right],$$

$$\big\|\nabla_\tau \nabla^m \Phi^1_{0Y}\big\|^2_{L^2} + \big\|\nabla^m \Phi^1_{0Y}\big\|^2_{H^1} \lesssim \big\|\mathcal{O}\big\|^2_{H^{m+1}} \textit{ for } \tau \in \left[\frac{1}{2}, 1\right],$$

$$\big\|\nabla_\tau \nabla^m \Phi^2_{0Y}\big\|^2_{L^2} + \big\|\nabla^m \Phi^2_{0Y}\big\|^2_{H^1} \lesssim \big\|\mathcal{O}\big\|^2_{H^{m+1+\eta}} \textit{ for } \tau \in (0, 1].$$

In particular, the singular component satisfies the following estimate for any $m < M$:

$$\big\|\nabla^m \Phi_{0Y}\big\|^2_{H^1} \lesssim \big(1 + |\log \tau|^2\big) \big\|\mathcal{O}\big\|^2_{H^{m+1+\eta}}. \tag{7.11}$$

Proof. We start with the estimate for $\nabla^m \Phi^2_{0Y}$. Using the standard ∇_τ multiplier in (7.4), we get for $\tau \in (0, 1]$:

$$\big\|\nabla_\tau \nabla^m \Phi^2_{0Y}\big\|^2_{L^2} + \big\|\nabla \nabla^m \Phi^2_{0Y}\big\|^2_{L^2} \lesssim \big\|\nabla \nabla^m \Phi^2_{0Y}\big\|^2_{L^2}\big|_{\tau=0} + \int_0^\tau \big\|\nabla_\tau \nabla^m \Phi^2_{0Y}\big\|_{L^2} \big\|\nabla \nabla^m \Phi^2_{0Y}\big\|_{L^2} +$$

$$+ \int_0^\tau \big\|\nabla \nabla^m \Phi^2_{0Y}\big\|_{L^2} \big\|[\nabla, \nabla_\tau] \nabla^m \Phi^2_{0Y}\big\|_{L^2} d\tau'.$$

We notice that due to Lemma 7.1, we have the terms $\int_0^\tau \tau' \|\nabla \nabla^m \Phi^2_{0Y}\|^2_{L^2}$ and $\int_0^\tau \tau' \|\nabla_\tau \nabla^m \Phi^2_{0Y}\|^2_{L^2}$ on the RHS, but these can be bounded using Grönwall. By the commutation formula (6.12), Lemma 6.7, and Grönwall, we get:

$$\big\|\nabla_\tau \nabla^m \Phi^2_{0Y}\big\|^2_{L^2} + \big\|\nabla \nabla^m \Phi^2_{0Y}\big\|^2_{L^2} \lesssim \big\|\mathcal{O}\big\|^2_{H^{m+1+\eta}} + \int_0^\tau \big\|\nabla^m \Phi^2_{0Y}\big\|^2_{L^2} d\tau'.$$

On the other hand, we also have the estimate:

$$\big\|\nabla^m\Phi^2_{0Y}\big\|^2_{L^2} \lesssim \big\|\mathcal{O}\big\|^2_{H^{m+\eta}} + \int_0^\tau \big\|\nabla^m\Phi^2_{0Y}\big\|_{L^2}\big\|\nabla_\tau\nabla^m\Phi^2_{0Y}\big\|_{L^2}d\tau' + \int_0^\tau \big\|\nabla^m\Phi^2_{0Y}\big\|^2_{L^2}d\tau'$$
$$\lesssim \big\|\mathcal{O}\big\|^2_{H^{m+\eta}} + \int_0^\tau \big\|\nabla_\tau\nabla^m\Phi^2_{0Y}\big\|^2_{L^2}d\tau'.$$

Combining the last two bounds, we obtain the desired estimate for $\big(\nabla^m\Phi_0\big)^2_Y$.

We notice that $\nabla^m\Phi^1_{0Y}$ satisfies the equation:

$$\nabla_\tau\bigg(\nabla_\tau\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg) + \frac{1}{\tau}\bigg(1+\frac{2}{\log\tau}\bigg)\nabla_\tau\frac{\nabla^m\Phi^1_{0Y}}{\log\tau} - 4\Delta\frac{\nabla^m\Phi^1_{0Y}}{\log\tau} = \psi\nabla\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}.$$

We use ∇_τ as a multiplier to obtain the estimate for any $\tau\in(0,1/2]$:

$$\bigg\|\nabla_\tau\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|^2_{L^2} + \bigg\|\nabla\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|^2_{L^2} \lesssim \bigg\|\nabla\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|^2_{L^2}\bigg|_{\tau=0} + \int_0^\tau \frac{\mathbf{1}_{[1/10,1/2]}}{\tau'|\log\tau'|}\bigg\|\nabla_\tau\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|^2_{L^2}d\tau' +$$
$$+\int_0^\tau\bigg\|\nabla_\tau\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|_{L^2}\bigg\|\nabla\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|_{L^2}d\tau' + \int_0^\tau\bigg\|\nabla\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|_{L^2}\bigg\|[\nabla,\nabla_\tau]\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|_{L^2}d\tau'.$$

We point out that the error term $\frac{1}{\tau'|\log\tau'|}\big\|\nabla_\tau\nabla^m\Phi^1_{0Y}/\log\tau\big\|^2_{L^2}\cdot\mathbf{1}_{[1/10,1/2]}$ appears on the right hand side because for $\tau\in[0,1/10]$ we have $1+2/\log\tau\gtrsim 1$, so the bulk term has a favorable sign. The error terms are estimated as usual, and we use Grönwall to obtain:

$$\bigg\|\nabla_\tau\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|^2_{L^2} + \bigg\|\nabla\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|^2_{L^2} \lesssim \big\|\mathcal{O}\big\|^2_{H^{m+1}} + \int_0^\tau\bigg\|\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|^2_{L^2}d\tau'.$$

We also have the estimate:

$$\bigg\|\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|^2_{L^2} \lesssim \big\|\mathcal{O}\big\|^2_{H^m} + \int_0^\tau\bigg\|\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|_{L^2}\bigg\|\nabla_\tau\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|_{L^2} + \int_0^\tau\bigg\|\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|^2_{L^2}$$
$$\lesssim \big\|\mathcal{O}\big\|^2_{H^m} + \int_0^\tau\bigg\|\nabla_\tau\frac{\nabla^m\Phi^1_{0Y}}{\log\tau}\bigg\|^2_{L^2}.$$

Combining the last two bounds, we obtain the desired estimate for $\big(\nabla^m\Phi_0\big)^1_Y$ on $\tau\in(0,1/2]$.

In the case of $\nabla^m\Phi^1_{0Y}$ for $\tau\in[1/2,1]$, we use the same estimates as for $\nabla^m\Phi^2_{0Y}$, but with data at $\tau=1/2$. This allows us to obtain the desired estimates for $\nabla^m\Phi^1_{0Y}$ on $\tau\in[1/2,1]$. $\square$

Remark 7.1. *We notice that the only place in the above proof where we use an inequality that is not sharp is when bounding the* $(\log\nabla)$ *operator using Lemma 6.7. In order to prove top order estimates, we will take a different approach to avoid this issue in Section 7.4.*

Next, we prove the following lower order estimates for the regular components, establishing (1.39):

Proposition 7.3. *Set* $\eta = 1/10$. *For any* $m < M$, *we have the following estimate:*

$$\left\|\nabla_\tau \nabla^m \Phi_{0J}\right\|_{L^2}^2 + \left\|\nabla^m \Phi_{0J}\right\|_{H^1}^2 + \sum_{i=1}^{I} \left\|\nabla_\tau \nabla^m \Phi_i\right\|_{L^2}^2 + \sum_{i=1}^{I} \left\|\nabla^m \Phi_i\right\|_{H^1}^2 \lesssim$$

$$\lesssim \left\|\mathfrak{h}\right\|_{H^{m+1}}^2 + \sum_{i=1}^{I} \left\|\Phi_i^0\right\|_{H^{m+1}}^2 + \left\|\mathcal{O}\right\|_{H^{m+1+\eta}}^2 + \sum_{i=0}^{I} \int_0^\tau \left\|F_m^i\right\|_{L^2}^2 d\tau'.$$

Proof. Using ∇_τ as a multiplier in the equations satisfied by the regular quantities, we get:

$$\left\|\nabla_\tau \nabla^m \Phi_{0J}\right\|_{L^2}^2 + \left\|\nabla \nabla^m \Phi_{0J}\right\|_{L^2}^2 \lesssim$$

$$\lesssim \left\|\mathfrak{h}_m\right\|_{H^1}^2 + \sum_{i=0}^{I} \int_0^\tau \left\|F_m^i\right\|_{L^2}^2 + \sum_{i=1}^{I} \int_0^\tau \left\|\nabla^{m+1} \Phi_i\right\|_{L^2}^2 + \int_0^\tau \left\|\nabla^m \Phi_{0J}\right\|_{L^2}^2,$$

$$\sum_{i=1}^{I} \left\|\nabla_\tau \nabla^m \Phi_i\right\|_{L^2}^2 + \sum_{i=1}^{I} \left\|\nabla^{m+1} \Phi_i\right\|_{L^2}^2 \lesssim$$

$$\sum_{i=1}^{I} \left\|\Phi_i^0\right\|_{H^{m+1}}^2 + \sum_{i=0}^{I} \int_0^\tau \left\|F_m^i\right\|_{L^2}^2 + \int_0^\tau \left\|\nabla \nabla^m \Phi_{0Y}\right\|_{L^2}^2 + \sum_{i=1}^{I} \int_0^\tau \left\|\nabla^m \Phi_i\right\|_{L^2}^2 + \int_0^\tau \left\|\nabla \nabla^m \Phi_{0J}\right\|_{L^2}^2.$$

In the second estimate, we dropped the bulk term with a favorable sign obtained from (5.13).

Using the above two bounds, together with Lemma 7.2 and Grönwall, we obtain:

$$\left\|\nabla_\tau \nabla^m \Phi_{0J}\right\|_{L^2}^2 + \left\|\nabla \nabla^m \Phi_{0J}\right\|_{L^2}^2 + \sum_{i=1}^{I} \left\|\nabla_\tau \nabla^m \Phi_i\right\|_{L^2}^2 + \sum_{i=1}^{I} \left\|\nabla^{m+1} \Phi_i\right\|_{L^2}^2 \lesssim$$

$$\lesssim \left\|\mathfrak{h}\right\|_{H^{m+1}}^2 + \sum_{i=1}^{I} \left\|\Phi_i^0\right\|_{H^{m+1}}^2 + \left\|\mathcal{O}\right\|_{H^{m+1+\eta}}^2 +$$

$$+ \sum_{i=0}^{I} \int_0^\tau \left\|F_m^i\right\|_{L^2}^2 + \sum_{i=1}^{I} \int_0^\tau \left\|\nabla^m \Phi_i\right\|_{L^2}^2 + \int_0^\tau \left\|\nabla^m \Phi_{0J}\right\|_{L^2}^2.$$

We can estimate the last two error terms on the RHS as before, which implies the conclusion. ☐

In the above proof, we used the following bound on $\mathfrak{h}_m$:

Lemma 7.2. *We have the estimate for any* $m < M$:

$$\left\|\mathfrak{h}_m\right\|_{H^1} \lesssim \left\|\mathfrak{h}\right\|_{H^{m+1}} + \left\|\mathcal{O}\right\|_{H^m}.$$

Proof. We can write:

$$\mathfrak{h}_m = \nabla^m h - 2(\log \nabla) \nabla^m \mathcal{O} = \nabla^m \mathfrak{h} - 2[(\log \nabla), \nabla^m] \mathcal{O}.$$

Using (6.15) and (6.16) applied to the projection operator P_k^2, we get for all $k \geq 0$:

$$\big\|[\nabla^m, P_k^2]\mathcal{O}\big\|_{H^1} \lesssim 2^{-k} C\big(\|Ri\not{e}m_0\|_{H^m}\big) \cdot \big\|\mathcal{O}\big\|_{H^m}.$$

As a result, we have that:

$$\big\|[(\log \nabla), \nabla^m]\mathcal{O}\big\|_{H^1} \lesssim C\big(\|Ri\not{e}m_0\|_{H^m}\big) \cdot \big\|\mathcal{O}\big\|_{H^m} \sum_{k \geq 0} 2^{-k} \log 2^k.$$

Moreover, we notice that $\|Ri\not{e}m_0\|_{H^{M-1}} \leq C_1 \lesssim 1$ by our bounds on the background spacetime in Theorem 7.1. □

We combine the above results to obtain the following lower order version of (7.2):

Corollary 7.1. *Set $\eta = 1/10$. We have the lower order estimate for $\nabla^m \Phi_0$, with $m < M$:*

$$\big\|\nabla^m \Phi_0\big\|_{H^1}^2 \lesssim \big(1 + |\log \tau|^2\big) \big\|\mathcal{O}\big\|_{H^{m+1+\eta}}^2 + \big\|\mathfrak{h}\big\|_{H^{m+1}}^2 + \sum_{i=1}^{I} \big\|\Phi_i^0\big\|_{H^{m+1}}^2 + \sum_{i=0}^{I} \int_0^\tau \big\|F_m^i\big\|_{L^2}^2 d\tau'.$$

We also need estimates for the commutator term $\mathcal{C} = \big(\nabla^M \Phi_0\big)_Y - \nabla\big(\nabla^{M-1}\Phi_0\big)_Y$. To compute the equation satisfied by $\mathcal{C}$, we first need to commute the equation of $\big(\nabla^{M-1}\Phi_0\big)_Y$ using the commutation formula (6.12):

$$\nabla_\tau\big(\nabla_\tau \nabla\big(\nabla^{M-1}\Phi_0\big)_Y\big) + \frac{1}{\tau}\nabla_\tau \nabla\big(\nabla^{M-1}\Phi_0\big)_Y - 4\Delta\nabla\big(\nabla^{M-1}\Phi_0\big)_Y = \psi \nabla\nabla\big(\nabla^{M-1}\Phi_0\big)_Y +$$
$$+ O\Big(\big|\big(\nabla^{M-1}\Phi_0\big)_Y\big| + \big|\nabla\big(\nabla^{M-1}\Phi_0\big)_Y\big| + \big|\tau\nabla_\tau\big(\nabla^{M-1}\Phi_0\big)_Y\big| + \big|\tau\nabla_\tau\nabla\big(\nabla^{M-1}\Phi_0\big)_Y\big|\Big),$$

where we used (5.15) and (5.16). As a result, we obtain that $\mathcal{C}$ satisfies the equation:

$$\nabla_\tau\big(\nabla_\tau \mathcal{C}\big) + \frac{1}{\tau}\nabla_\tau \mathcal{C} - 4\Delta\mathcal{C} = \psi\nabla\mathcal{C} + O\Big(\big|\tau\nabla_\tau\mathcal{C}\big|\Big) +$$
$$+ O\Big(\big|\big(\nabla^{M-1}\Phi_0\big)_Y\big| + \big|\nabla\big(\nabla^{M-1}\Phi_0\big)_Y\big| + \big|\tau\nabla_\tau\big(\nabla^{M-1}\Phi_0\big)_Y\big| + \big|\tau\nabla_\tau\big(\nabla^{M}\Phi_0\big)_Y\big|\Big).$$

Moreover, we notice that $\mathcal{C}$ is a regular quantity at $\tau = 0$, satisfying the expansion:

$$\mathcal{C} = 2[\log\nabla, \nabla]\nabla^{M-1}\mathcal{O} + O\big(\tau^2|\log(\tau)|^2\big),\ \nabla_\tau\mathcal{C} = O\Big(\tau|\log(\tau)|^2\Big).$$

Note that as in the proof of Lemma 7.2 we get:

$$\big\|[\log\nabla, \nabla]\nabla^{M-1}\mathcal{O}\big\|_{H^1} \lesssim \big\|\mathcal{O}\big\|_{H^M}.$$

We conclude the section by proving the estimate:

Proposition 7.4. *Set $\eta = 1/10$. The commutator term $\mathcal{C} = \big(\nabla^M \Phi_0\big)_Y - \nabla\big(\nabla^{M-1}\Phi_0\big)_Y$ satisfies the estimate:*

$$\big\|\nabla_\tau \mathcal{C}\big\|_{L^2}^2 + \big\|\mathcal{C}\big\|_{H^1}^2 \lesssim \big\|\mathcal{O}\big\|_{H^{M+\eta}}^2 + \int_0^\tau \big\|\tau' \nabla_\tau \nabla^M \Phi_{0Y}\big\|_{L^2}^2 d\tau'. \tag{7.12}$$

Additionally, for any $k \geq 0$ and $\tau \in (0, 2^{-k-1}]$ we have:

$$\big\|\mathcal{C}\big\|_{L^2}^2 \lesssim \big\|\mathcal{O}\big\|_{H^{M+\eta}}^2 + 2^{-k} \int_0^\tau \big\|\tau' \nabla_\tau \nabla^M \Phi_{0Y}\big\|_{L^2}^2 d\tau'. \tag{7.13}$$

Proof. Using the standard ∇_τ multiplier and the previous lower order estimates we get:

$$\big\|\nabla_\tau \mathcal{C}\big\|_{L^2}^2 + \big\|\mathcal{C}\big\|_{H^1}^2 \lesssim$$

$$\lesssim \big\|\mathcal{O}\big\|_{H^M}^2 + \int_0^\tau \big\|\nabla^{M-1}\Phi_{0Y}\big\|_{H^1}^2 + \int_0^\tau \big\|\tau' \nabla_\tau \nabla^{M-1} \Phi_{0Y}\big\|_{L^2}^2 + \int_0^\tau \big\|\tau' \nabla_\tau \nabla^M \Phi_{0Y}\big\|_{L^2}^2$$

$$\lesssim \big\|\mathcal{O}\big\|_{H^{M+\eta}}^2 + \int_0^\tau \big\|\tau' \nabla_\tau \nabla^M \Phi_{0Y}\big\|_{L^2}^2$$

As before, we can also estimate $\big\|\mathcal{C}\big\|_{L^2}^2$ in order to obtain the conclusion. ☐

7.2.2 Fractional Estimates

In this section, we prove estimates in $H^{1/2}$ for $\nabla(\nabla^{M-1}\Phi_0)_Y$ and $\nabla_\tau(\nabla^{M-1}\Phi_0)_Y$. As explained in Section 1.3.3, the error terms obtained in Section 7.3 in the proof of the top order estimates for the singular component $\big(\nabla^M \Phi_0\big)_Y$ can be simplified significantly using the fractional estimates proved in this section.

For the rest of the section we prove the following result:

Proposition 7.5. *Set $\eta = 1/10$. The singular component $(\nabla^{M-1}\Phi_0)_Y$ satisfies the estimates:*

$$\big\|\nabla(\nabla^{M-1}\Phi_0)_Y\big\|_{H^{1/2}}^2 \lesssim \big(1 + |\log \tau|^2\big)\big\|\mathcal{O}\big\|_{H^{M+1/2+\eta}}^2$$

$$\big\|\nabla_\tau(\nabla^{M-1}\Phi_0)_Y\big\|_{H^{1/2}}^2 \lesssim \frac{1}{\tau^2}\big\|\mathcal{O}\big\|_{H^{M+1/2+\eta}}^2.$$

In order to prove this result we treat separately the components $\nabla^{M-1}\Phi_{0Y}^1$ and $\nabla^{M-1}\Phi_{0Y}^2$. The above fractional estimates will be a consequence of Propositions 7.6 and 7.7. We start with the component $\mathcal{A} := \nabla^{M-1}\Phi_{0Y}^2$, defined using Proposition 7.1 with asymptotic data $\big(0, 2(\log\nabla)\nabla^{M-1}\mathcal{O}\big)$. Thus, $\mathcal{A}$ satisfies the equation:

$$\nabla_\tau\big(\nabla_\tau \mathcal{A}\big) + \frac{1}{\tau}\nabla_\tau \mathcal{A} - 4\Delta\mathcal{A} = \psi\nabla\mathcal{A},$$

and the following asymptotic expansions hold as $\tau \to 0$:

$$\mathcal{A}(\tau) = 2(\log \nabla)\nabla^{M-1}\mathcal{O} + O\big(\tau^2|\log(\tau)|^2\big),\ \nabla_\tau\mathcal{A}(\tau) = O\big(\tau|\log(\tau)|^2\big).$$

Proposition 7.6. *Set* $\eta = 1/10$. $\mathcal{A}$ *satisfies the estimate:*

$$\big\|\nabla_\tau\mathcal{A}\big\|^2_{H^{1/2}} + \big\|\nabla\mathcal{A}\big\|^2_{H^{1/2}} \lesssim \big\|\mathcal{O}\big\|^2_{H^{M+1/2+\eta}}.$$

Proof. For any $k \geq 0$, we apply P_k to the equation satisfied by $\mathcal{A}$ to get:

$$\nabla_\tau\big(P_k\nabla_\tau\mathcal{A}\big) + \frac{1}{\tau}P_k\nabla_\tau\mathcal{A} - 4\Delta P_k\mathcal{A} = \psi\nabla P_k\mathcal{A} + [P_k,\psi]\nabla\mathcal{A} + \psi[P_k,\nabla]\mathcal{A} + [\nabla_\tau, P_k]\nabla_\tau\mathcal{A}.$$

We contract this equation with $P_k\nabla_\tau\mathcal{A}$ and integrate by parts to obtain the energy estimate:

$$\begin{aligned}\big\|P_k\nabla_\tau\mathcal{A}\big\|^2_{L^2} + \big\|\nabla P_k\mathcal{A}\big\|^2_{L^2} &\lesssim \big\|\nabla P_k\mathcal{A}|_{\tau=0}\big\|^2_{L^2} + \int_0^\tau \tau'\big\|\nabla P_k\mathcal{A}\big\|_{L^2}\cdot\big\|\nabla[P_k,\nabla_4]\mathcal{A}\big\|_{L^2} +\\ &+ \int_0^\tau \tau'\big\|\nabla P_k\mathcal{A}\big\|_{L^2}\cdot\big\|[\nabla,\nabla_4]P_k\mathcal{A}\big\|_{L^2} + \int_0^\tau \big\|\nabla P_k\mathcal{A}\big\|_{L^2}\cdot\big\|P_k\nabla_\tau\mathcal{A}\big\|_{L^2} +\\ &+ \int_0^\tau \big\|[P_k,\psi]\nabla\mathcal{A}\big\|_{L^2}\cdot\big\|P_k\nabla_\tau\mathcal{A}\big\|_{L^2} + \int_0^\tau \big\|[P_k,\nabla]\mathcal{A}\big\|_{L^2}\cdot\big\|P_k\nabla_\tau\mathcal{A}\big\|_{L^2} +\\ &+ \int_0^\tau \tau'\big\|[\nabla_4,P_k]\nabla_\tau\mathcal{A}\big\|_{L^2}\cdot\big\|P_k\nabla_\tau\mathcal{A}\big\|_{L^2}.\end{aligned}$$

We use Grönwall:

$$\begin{aligned}\big\|P_k\nabla_\tau\mathcal{A}\big\|^2_{L^2} + \big\|\nabla P_k\mathcal{A}\big\|^2_{L^2} &\lesssim \big\|\nabla P_k\mathcal{A}|_{\tau=0}\big\|^2_{L^2} + \int_0^\tau \big\|\nabla[P_k,\nabla_4]\mathcal{A}\big\|^2_{L^2} +\\ &+ \int_0^\tau \big\|[\nabla,\nabla_4]P_k\mathcal{A}\big\|^2_{L^2} + \int_0^\tau \big\|[P_k,\psi]\nabla\mathcal{A}\big\|^2_{L^2} + \int_0^\tau \big\|[P_k,\nabla]\mathcal{A}\big\|^2_{L^2} + \int_0^\tau \big\|[\nabla_4,P_k]\nabla_\tau\mathcal{A}\big\|^2_{L^2}.\end{aligned}$$

We use the bounds in Lemma 6.3 and Lemma 6.4 to control the commutation terms:

$$\begin{aligned}\big\|P_k\nabla_\tau\mathcal{A}\big\|^2_{L^2} + \big\|\nabla P_k\mathcal{A}\big\|^2_{L^2} &\lesssim \big\|\nabla P_k\mathcal{A}|_{\tau=0}\big\|^2_{L^2} + \int_0^\tau \big\|\widetilde{\underline{P}}_k\nabla\mathcal{A}\big\|^2_{L^2} + \int_0^\tau 2^{-2k}\big\|\mathcal{A}\big\|^2_{H^1}\\ &+ \int_0^\tau \big\|P_k\mathcal{A}\big\|^2_{H^1} + \int_0^\tau \big\|\widetilde{\underline{P}}_k\nabla_\tau\mathcal{A}\big\|^2_{L^2} + \int_0^\tau 2^{-2k}\big\|\nabla_\tau\mathcal{A}\big\|^2_{L^2}.\end{aligned}$$

As a result, we get that:

$$\begin{aligned}\big\|P_k\nabla_\tau\mathcal{A}\big\|^2_{L^2} + \big\|P_k\nabla\mathcal{A}\big\|^2_{L^2} &\lesssim \big\|P_k\nabla\mathcal{A}|_{\tau=0}\big\|^2_{L^2} + 2^{-2k}\big\|\mathcal{A}|_{\tau=0}\big\|^2_{L^2} + \int_0^\tau \big\|\widetilde{\underline{P}}_k\nabla\mathcal{A}\big\|^2_{L^2} +\\ &+ \int_0^\tau 2^{-2k}\big\|\mathcal{A}\big\|^2_{H^1} + \int_0^\tau \big\|P_k\mathcal{A}\big\|^2_{L^2} + \int_0^\tau \big\|\widetilde{\underline{P}}_k\nabla_\tau\mathcal{A}\big\|^2_{L^2} + \int_0^\tau 2^{-2k}\big\|\nabla_\tau\mathcal{A}\big\|^2_{L^2}.\end{aligned}$$

We multiply by 2^k and sum over all $k \geq 0$. This amounts to taking half of a derivative.

$$\sum_{k\geq 0} 2^k\Big(\big\|P_k\nabla_\tau\mathcal{A}\big\|^2_{L^2} + \big\|P_k\nabla\mathcal{A}\big\|^2_{L^2}\Big) \lesssim$$

$$\lesssim \big\|\mathcal{A}|_{\tau=0}\big\|_{L^2}^2 + \int_0^\tau \big\|\mathcal{A}\big\|_{H^1}^2 + \int_0^\tau \big\|\nabla_\tau \mathcal{A}\big\|_{L^2}^2 + \sum_{k\geq 0} 2^k \big\|P_k \nabla \mathcal{A}|_{\tau=0}\big\|_{L^2}^2 +$$

$$+ \int_0^\tau \sum_{k\geq 0} 2^k \big\|\underline{\widetilde{P}}_k \nabla \mathcal{A}\big\|_{L^2}^2 + \int_0^\tau \sum_{k\geq 0} 2^k \big\|P_k \mathcal{A}\big\|_{L^2}^2 + \int_0^\tau \sum_{k\geq 0} 2^k \big\|\underline{\widetilde{P}}_k \nabla_\tau \mathcal{A}\big\|_{L^2}^2.$$

We recall that from the standard lower order estimates we have:

$$\big\|\nabla_\tau \mathcal{A}\big\|_{L^2}^2 + \big\|\mathcal{A}\big\|_{H^1}^2 \lesssim \big\|\mathcal{O}\big\|_{H^{M+\eta}}^2.$$

Since $\sum_k P_k^2 = I$, we obtain using our definition of fractional Sobolev spaces in Section 6:

$$\big\|\nabla_\tau \mathcal{A}\big\|_{H^{1/2}}^2 + \big\|\nabla \mathcal{A}\big\|_{H^{1/2}}^2 \lesssim \big\|\mathcal{O}\big\|_{H^{M+\eta}}^2 + \big\|\nabla \mathcal{A}|_{\tau=0}\big\|_{H^{1/2}}^2 + \int_0^\tau \big\|\nabla_\tau \mathcal{A}\big\|_{H^{1/2}}^2 + \big\|\nabla \mathcal{A}\big\|_{H^{1/2}}^2 d\tau'$$

$$\lesssim \big\|\mathcal{O}\big\|_{H^{M+1/2+\eta}}^2 + \int_0^\tau \big\|\nabla_\tau \mathcal{A}\big\|_{H^{1/2}}^2 + \big\|\nabla \mathcal{A}\big\|_{H^{1/2}}^2 d\tau'.$$

Finally, we obtain the desired conclusion by Grönwall. □

Next, we consider the component $\mathcal{B} := \nabla^{M-1}\Phi^1_{0Y}$, defined using Proposition 7.1 with asymptotic data $\big(2\nabla^{M-1}\mathcal{O}, 0\big)$. Thus, $\mathcal{B}$ satisfies the equation:

$$\nabla_\tau\big(\nabla_\tau \mathcal{B}\big) + \frac{1}{\tau}\nabla_\tau \mathcal{B} - 4\Delta \mathcal{B} = \psi \nabla \mathcal{B},$$

and the following asymptotic expansions hold as $\tau \to 0$:

$$\mathcal{B}(\tau) = 2\nabla^{M-1}\mathcal{O}\log(\tau) + O\big(\tau^2|\log(\tau)|^2\big),\ \nabla_\tau \mathcal{B}(\tau) = \frac{2\nabla^{M-1}\mathcal{O}}{\tau} + O\big(\tau|\log(\tau)|^2\big).$$

Proposition 7.7. *$\mathcal{B}$ satisfies the estimates:*

$$\big\|\nabla \mathcal{B}\big\|_{H^{1/2}}^2 \lesssim \big(1 + |\log \tau|^2\big)\big\|\mathcal{O}\big\|_{H^{M+1/2}}^2,$$

$$\big\|\nabla_\tau \mathcal{B}\big\|_{H^{1/2}}^2 \lesssim \frac{1}{\tau^2}\big\|\mathcal{O}\big\|_{H^{M+1/2}}^2.$$

Proof. We denote $\mathcal{D} = \mathcal{B}/\log\tau$. Then $\mathcal{D}$ satisfies the equation:

$$\nabla_\tau\big(\nabla_\tau \mathcal{D}\big) + \frac{1}{\tau}\left(1 + \frac{2}{\log \tau}\right)\nabla_\tau \mathcal{D} - 4\Delta \mathcal{D} = \psi \nabla \mathcal{D},$$

$$\mathcal{D}(\tau) = 2\nabla^{M-1}\mathcal{O} + O\big(\tau^2|\log(\tau)|^2\big),\ \nabla_\tau \mathcal{D}(\tau) = O\big(\tau|\log(\tau)|^2\big).$$

This has the same properties needed to do energy estimates as the equation satisfied by $\mathcal{A}$. Thus, an analogous proof gives for $\tau \in (0, 1/2]$ and any $k \geq 0$:

$$\big\|P_k \nabla_\tau \mathcal{D}\big\|_{L^2}^2 + \big\|\nabla P_k \mathcal{D}\big\|_{L^2}^2 \lesssim \big\|\nabla P_k \mathcal{D}|_{\tau=0}\big\|_{L^2}^2 + \int_0^\tau \big\|\underline{\widetilde{P}}_k \nabla \mathcal{D}\big\|_{L^2}^2 + \int_0^\tau 2^{-2k}\big\|\mathcal{D}\big\|_{H^1}^2$$

$$+\int_0^\tau \big\|P_k\mathcal{D}\big\|^2_{H^1}+\int_0^\tau \big\|\underline{\widetilde{P}}_k\nabla_\tau\mathcal{D}\big\|^2_{L^2}+\int_0^\tau 2^{-2k}\big\|\nabla_\tau\mathcal{D}\big\|^2_{L^2}+\int_0^\tau \frac{\mathbf{1}_{[1/10,1/2]}}{\tau'|\log\tau'|}\big\|P_k\nabla_\tau\mathcal{D}\big\|^2_{L^2}d\tau'.$$

We point out that the error term $\frac{1}{\tau'|\log\tau'|}\big\|P_k\nabla_\tau\mathcal{D}\big\|^2_{L^2}\cdot\mathbf{1}_{[1/10,1/2]}$ appears on the RHS because for $\tau\in[0,1/10]$ we have $1+2/\log\tau\gtrsim 1$. This can be estimated as usual using Grönwall. We multiply by 2^k and sum over k :

$$\sum_{k\geq 0}2^k\Big(\big\|P_k\nabla_\tau\mathcal{D}\big\|^2_{L^2}+\big\|P_k\nabla\mathcal{D}\big\|^2_{L^2}\Big)\lesssim$$

$$\lesssim\big\|\mathcal{D}|_{\tau=0}\big\|^2_{L^2}+\int_0^\tau\big\|\mathcal{D}\big\|^2_{H^1}+\int_0^\tau\big\|\nabla_\tau\mathcal{D}\big\|^2_{L^2}+\sum_{k\geq 0}2^k\big\|P_k\nabla\mathcal{D}|_{\tau=0}\big\|^2_{L^2}+$$

$$+\int_0^\tau\sum_{k\geq 0}2^k\big\|\underline{\widetilde{P}}_k\nabla\mathcal{D}\big\|^2_{L^2}+\int_0^\tau\sum_{k\geq 0}2^k\big\|P_k\mathcal{D}\big\|^2_{L^2}+\int_0^\tau\sum_{k\geq 0}2^k\big\|\underline{\widetilde{P}}_k\nabla_\tau\mathcal{D}\big\|^2_{L^2}.$$

We recall that from the standard lower order estimates we have for $\tau\in(0,1/2]$:

$$\big\|\nabla_\tau\mathcal{D}\big\|^2_{L^2}+\big\|\mathcal{D}\big\|^2_{H^1}\lesssim\big\|\mathcal{O}\big\|^2_{H^M}.$$

Since $\sum_k P_k^2=I$, we obtain using our definition of fractional Sobolev spaces in Section 6:

$$\big\|\nabla_\tau\mathcal{D}\big\|^2_{H^{1/2}}+\big\|\nabla\mathcal{D}\big\|^2_{H^{1/2}}\lesssim\big\|\mathcal{O}\big\|^2_{H^M}+\big\|\nabla\mathcal{D}|_{\tau=0}\big\|^2_{H^{1/2}}+\int_0^\tau\big\|\nabla_\tau\mathcal{D}\big\|^2_{H^{1/2}}+\big\|\nabla\mathcal{D}\big\|^2_{H^{1/2}}d\tau'$$

$$\lesssim\big\|\mathcal{O}\big\|^2_{H^{M+1/2}}+\int_0^\tau\big\|\nabla_\tau\mathcal{D}\big\|^2_{H^{1/2}}+\big\|\nabla\mathcal{D}\big\|^2_{H^{1/2}}d\tau'.$$

We obtain by Grönwall that for any $\tau\in(0,1/2]$:

$$\big\|\nabla_\tau\mathcal{D}\big\|^2_{H^{1/2}}+\big\|\nabla\mathcal{D}\big\|^2_{H^{1/2}}\lesssim\big\|\mathcal{O}\big\|^2_{H^{M+1/2}}.$$

This implies that for any $\tau\in(0,1/2]$:

$$\big\|\nabla\mathcal{B}\big\|^2_{H^{1/2}}\lesssim|\log\tau|^2\big\|\mathcal{O}\big\|^2_{H^{M+1/2}},$$

$$\big\|\nabla_\tau\mathcal{B}\big\|^2_{H^{1/2}}\lesssim|\log\tau|^2\big\|\mathcal{O}\big\|^2_{H^{M+1/2}}+\frac{1}{\tau^2|\log\tau|^2}\big\|\mathcal{B}\big\|^2_{H^1}\lesssim|\log\tau|^2\big\|\mathcal{O}\big\|^2_{H^{M+1/2}}+\frac{1}{\tau^2}\big\|\mathcal{O}\big\|^2_{H^M}.$$

In particular, this also implies that:

$$\big\|\nabla_\tau\mathcal{B}\big\|^2_{H^{1/2}}\big|_{\tau=1/2}+\big\|\nabla\mathcal{B}\big\|^2_{H^{1/2}}\big|_{\tau=1/2}\lesssim\big\|\mathcal{O}\big\|^2_{H^{M+1/2}}.$$

Using the equation for $\mathcal{B}$ on the time interval $\tau\in[1/2,1]$, we repeat the energy estimate that we did for $\mathcal{A}$:

$$\sum_{k\geq 0}2^k\Big(\big\|P_k\nabla_\tau\mathcal{B}\big\|^2_{L^2}+\big\|P_k\nabla\mathcal{B}\big\|^2_{L^2}\Big)\lesssim$$

$$\lesssim \big\|\mathcal{B}\big|_{\tau=\frac{1}{2}}\big\|^2_{L^2} + \int_{\frac{1}{2}}^{\tau} \big\|\mathcal{B}\big\|^2_{H^1} + \int_{\frac{1}{2}}^{\tau} \big\|\nabla_\tau\mathcal{B}\big\|^2_{L^2} + \sum_{k\geq 0} 2^k \big\|P_k\nabla\mathcal{B}\big|_{\tau=\frac{1}{2}}\big\|^2_{L^2} +$$

$$+ \sum_{k\geq 0} 2^k \big\|P_k\nabla_\tau\mathcal{B}\big|_{\tau=\frac{1}{2}}\big\|^2_{L^2} + \int_{\frac{1}{2}}^{\tau}\sum_{k\geq 0} 2^k \big\|\underline{\widetilde{P}}_k\nabla\mathcal{B}\big\|^2_{L^2} + \int_{\frac{1}{2}}^{\tau}\sum_{k\geq 0} 2^k \big\|P_k\mathcal{B}\big\|^2_{L^2} + \int_{\frac{1}{2}}^{\tau}\sum_{k\geq 0} 2^k \big\|\underline{\widetilde{P}}_k\nabla_\tau\mathcal{B}\big\|^2_{L^2}.$$

We recall that from the standard lower order estimates we have for $\tau \in [1/2, 1]$:

$$\big\|\nabla_\tau\mathcal{B}\big\|^2_{L^2} + \big\|\mathcal{B}\big\|^2_{H^1} \lesssim \big\|\mathcal{O}\big\|^2_{H^M}.$$

As a result, we have that for $\tau \in [1/2, 1]$:

$$\big\|\nabla_\tau\mathcal{B}\big\|^2_{H^{1/2}} + \big\|\nabla\mathcal{B}\big\|^2_{H^{1/2}} \lesssim \big\|\mathcal{O}\big\|^2_{H^{M+1/2}} + \int_{\frac{1}{2}}^{\tau} \big\|\nabla_\tau\mathcal{B}\big\|^2_{H^{1/2}} + \int_{\frac{1}{2}}^{\tau} \big\|\nabla\mathcal{B}\big\|^2_{H^{1/2}}.$$

We combine the estimates for $\tau \in (0, 1/2]$ and $\tau \in [1/2, 1]$ to obtain the conclusion. □

7.3 Top Order Estimates for the Singular Component

To prove top order estimates, we need a precise understanding of the behavior of the P_k projections of $\big(\nabla^M\Phi_0\big)_Y$. As in the case of the linear wave equation on de Sitter background studied in [Cic23], we need to treat differently the low frequency regime $\tau \in (0, 2^{-k-1}]$ and the high frequency regime $\tau \in [2^{-k-1}, 1]$.

7.3.1 Low Frequency Regime Estimates

For every $k \geq 0$ we have the following expansions:

$$P_k\big(\nabla^M\Phi_0\big)_Y(\tau) = 2P_k\nabla^M\mathcal{O}\log(2^k\tau) + R_k\nabla^M\mathcal{O} + O\big(\tau^2|\log(\tau)|^2\big), \tag{7.14}$$

$$P_k\nabla_{2^{-k}\partial_\tau}\big(\nabla^M\Phi_0\big)_Y(\tau) = 2P_k\nabla^M\mathcal{O}\frac{1}{2^k\tau} + O\big(\tau|\log(\tau)|^2\big), \tag{7.15}$$

where we defined $R_k\nabla^M\mathcal{O}$ by (6.30). We point out that $\nabla^M\mathcal{O}$, $P_k\nabla^M\mathcal{O}$, $R_k\nabla^M\mathcal{O}$ on the right hand side of (7.14)-(7.15) are defined at $\tau = 0$ and extended by Lie transport. This can be done since the difference between projecting with respect to $\slashed{g}_0$ or $\slashed{g}_\tau$ is $O(\tau^2)$ according to Lemma 6.9.

We prove the main low frequency regime estimates in Proposition 7.9. Our goal is to prove energy estimates on $\tau \in (0, 2^{-k-1}]$ for the singular component $P_k\big(\nabla^M\Phi_0\big)_Y/\log(2^k\tau)$, renormalized to account for the contribution of $R_k\nabla^M\mathcal{O}$. According to the expansions (7.14)-(7.15), the asymptotic data will be given by $2P_k\nabla^M\mathcal{O}$.

It is convenient to consider the new time variable $t = 2^k\tau$. When using ∇_t below as a multiplier, we can control the error terms resulting from time derivatives of the volume form and apply Gronwall on the interval $t \in (0, 2^k]$ since for any tensor F we have:

$$\frac{1}{2}\frac{d}{dt}\|F\|_{L^2}^2 = \int_{S_t} F \cdot \nabla_t F + O\big(2^{-2k} t \|F\|_{L^2}^2\big). \tag{7.16}$$

We prove a preliminary low frequency regime estimate:

Proposition 7.8. *For any $k \geq 0$ and $\tau \leq 2^{-k-1}$, we have that $\big(\nabla^M \Phi_0\big)_Y$ satisfies the estimate:*

$$\left\| P_k \nabla_\tau \frac{\big(\nabla^M \Phi_0\big)_Y}{\log 2^k\tau} - R_k \nabla^M \mathcal{O} \partial_\tau \bigg(\frac{1}{\log 2^k \tau}\bigg)\right\|_{L^2}^2 + \left\| \nabla \bigg(P_k \frac{\big(\nabla^M \Phi_0\big)_Y}{\log 2^k\tau} - \frac{R_k \nabla^M \mathcal{O}}{\log 2^k \tau}\bigg)\right\|_{L^2}^2 \lesssim$$

$$\lesssim \big\|\nabla P_k \nabla^M \mathcal{O}\big\|_{L^2}^2 + \big\|\underline{P}_k \nabla^M \mathcal{O}\big\|_{H^1}^2 + \int_0^\tau \frac{(\tau')^2}{2^k} \left\| \nabla [P_k, \nabla_4] \frac{\big(\nabla^M \Phi_0\big)_Y}{\log 2^k \tau'}\right\|_{L^2}^2 +$$

$$+ \int_0^\tau \frac{(\tau')^2}{2^k} \left\| [P_k, \nabla_4] \nabla_\tau \frac{\big(\nabla^M \Phi_0\big)_Y}{\log 2^k \tau'}\right\|_{L^2}^2 + \int_0^\tau \left\| \frac{\big(\nabla^M \Phi_0\big)_Y}{\log 2^k \tau'}\right\|_{L^2}^2.$$

Proof. We first rewrite the equation (7.4) using the time variable $t = 2^k \tau \leq 1/2$. We have that:

$$\nabla_t\big(\nabla_t\big(\nabla^m \Phi_0\big)_Y\big) + \frac{1}{t}\nabla_t\big(\nabla^m \Phi_0\big)_Y - \frac{4}{2^{2k}}\Delta\big(\nabla^m \Phi_0\big)_Y = \frac{1}{2^{2k}}\psi \nabla \big(\nabla^m \Phi_0\big)_Y.$$

Moreover, we compute that for any tensor F we have:

$$\nabla_t\bigg(\nabla_t \frac{F}{\log t}\bigg) + \frac{1}{t}\bigg(1 + \frac{2}{\log t}\bigg)\nabla_t \frac{F}{\log t} = \frac{1}{\log t}\nabla_t\big(\nabla_t F\big) + \frac{1}{t \log t}\nabla_t F.$$

As a result, we can rewrite (7.4) as:

$$\nabla_t\bigg(\nabla_t \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t}\bigg) + \frac{1}{t}\bigg(1 + \frac{2}{\log t}\bigg)\nabla_t \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t} - \frac{4}{2^{2k}}\Delta \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t} = \frac{1}{2^{2k}}\psi \nabla \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t}. \tag{7.17}$$

We apply P_k to the equation (7.17):

$$\nabla_t\bigg(P_k \nabla_t \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t}\bigg) + \frac{1}{t}\bigg(1 + \frac{2}{\log t}\bigg)P_k \nabla_t \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t} - \frac{4}{2^{2k}}\Delta P_k \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t} =$$

$$= \frac{1}{2^{2k}}\psi \nabla P_k \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t} + [\nabla_t, P_k]\nabla_t \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t} + \frac{1}{2^{2k}}[P_k, \nabla]\frac{\psi\big(\nabla^M \Phi_0\big)_Y}{\log t}$$

$$- \frac{1}{2^{2k}}P_k\bigg(\nabla\psi \cdot \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t}\bigg) + \frac{1}{2^{2k}}[\nabla P_k, \psi]\frac{\big(\nabla^M \Phi_0\big)_Y}{\log t}.$$

We introduce the notation:

$$X = P_k \nabla_t \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t} - R_k \nabla^M \mathcal{O} \partial_t\bigg(\frac{1}{\log t}\bigg),\ Y = P_k \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t} - \frac{R_k \nabla^M \mathcal{O}}{\log t},$$

$$Z = 2^{-2k}\frac{4\Delta R_k \nabla^M \mathcal{O} + \psi \nabla R_k \nabla^M \mathcal{O}}{\log t} + \frac{\nabla_t\big(R_k \nabla^M \mathcal{O}\big)}{t|\log t|^2}.$$

Our definitions imply the following relation between X and Y :

$$X = \nabla_t Y + [P_k, \nabla_t] \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t} + \frac{\nabla_t R_k \nabla^M \mathcal{O}}{\log t}. \tag{7.18}$$

Using the notation introduced, we rewrite the above equation as:

$$\nabla_t X + \frac{1}{t}\Big(1 + \frac{2}{\log t}\Big) X - \frac{4}{2^{2k}} \Delta Y = Err, \tag{7.19}$$

where we denote the error terms:

$$Err = \frac{1}{2^{2k}} \psi \nabla Y + [\nabla_t, P_k] \nabla_t \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t} + \frac{1}{2^{2k}} [P_k, \nabla] \frac{\psi \big(\nabla^M \Phi_0\big)_Y}{\log t} \tag{7.20}$$

$$+ Z - \frac{1}{2^{2k}} P_k \Big(\nabla \psi \cdot \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t} \Big) + \frac{1}{2^{2k}} [\nabla P_k, \psi] \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t}. \tag{7.21}$$

To obtain an energy estimate, we contract (7.19) with X and use (7.16), (7.18) to get:

$$\frac{1}{2}\frac{d}{dt} \|X\|_{L^2}^2 + \frac{1}{t}\Big(1 + \frac{2}{\log t}\Big) \|X\|_{L^2}^2 - \frac{4}{2^{2k}} \int_{S_t} \Delta Y \cdot \Big(\nabla_t Y + [P_k, \nabla_t] \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t} + \frac{\nabla_t R_k \nabla^M \mathcal{O}}{\log t} \Big) =$$

$$= \int_{S_t} X \cdot Err + O\big(2^{-2k} t \|X\|_{L^2}^2\big).$$

We use the divergence theorem for the ΔY term, then use (7.16) to integrate in time, also using (7.14)-(7.15) to compute the boundary terms at $t = 0$:

$$\|X\|_{L^2}^2 + \int_0^t \frac{2}{t'}\Big(1 + \frac{2}{\log t'}\Big) \|X\|_{L^2}^2 + \frac{4}{2^{2k}} \|\nabla Y\|_{L^2}^2 \lesssim$$

$$\lesssim \frac{1}{2^{2k}} \|\nabla P_k \nabla^M \mathcal{O}\|_{L^2}^2 + \int_0^t \frac{1}{2^{2k}} \|\nabla Y\|_{L^2} \Big\| \nabla [P_k, \nabla_t] \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t'} \Big\|_{L^2} +$$

$$+ \int_0^t \frac{1}{2^{2k}} \|\nabla Y\|_{L^2} \|[\nabla, \nabla_t] Y\|_{L^2} + \int_0^t \frac{1}{2^{2k}} \|\nabla Y\|_{L^2} \Big\| \frac{\nabla \nabla_t R_k \nabla^M \mathcal{O}}{\log t'} \Big\|_{L^2} +$$

$$+ \int_0^t \|X\|_{L^2} \|Err\|_{L^2} + \int_0^t 2^{-2k} t' \|X\|_{L^2}^2 + \int_0^t 2^{-4k} t' \|\nabla Y\|_{L^2}^2.$$

We notice that $1 + 2/\log t \gtrsim 1$ for $t \in [0, 1/10]$, but this quantity changes sign for $t \in [1/10, 1/2]$. Thus, we can drop the second term on the LHS, at the cost of introducing the error term $\int_0^t \|X\|_{L^2}^2 dt'$ on the RHS. Using Gronwall, the definition of the error term in (7.20)-(7.21), and the fact that $\nabla_t = t \cdot 2^{-2k+1} \nabla_4$, we obtain the energy estimate:

$$\|X\|_{L^2}^2 + \frac{1}{2^{2k}} \|\nabla Y\|_{L^2}^2 \lesssim \frac{1}{2^{2k}} \|\nabla P_k \nabla^M \mathcal{O}\|_{L^2}^2 + \int_0^t \frac{(t')^2}{2^{6k}} \Big\| \nabla [P_k, \nabla_4] \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t'} \Big\|_{L^2}^2 +$$

$$+ \int_0^t \frac{(t')^2}{2^{6k}} \|[\nabla, \nabla_4] Y\|_{L^2}^2 + \int_0^t \frac{1}{2^{2k}} \Big\| \frac{\nabla \nabla_t R_k \nabla^M \mathcal{O}}{\log t'} \Big\|_{L^2}^2 + \int_0^t \frac{(t')^2}{2^{4k}} \Big\| [\nabla_4, P_k] \nabla_t \frac{\big(\nabla^M \Phi_0\big)_Y}{\log t'} \Big\|_{L^2}^2 +$$

$$\int_0^t \|Z\|_{L^2}^2 + \int_0^t \frac{1}{2^{4k}} \left\| [P_k,\nabla] \frac{\psi\big(\nabla^M\Phi_0\big)_Y}{\log t'}\right\|_{L^2}^2 +$$
$$+ \int_0^t \frac{1}{2^{4k}}\left\|P_k\Big(\nabla\psi\cdot \frac{\big(\nabla^M\Phi_0\big)_Y}{\log t'}\Big)\right\|_{L^2}^2 + \int_0^t \frac{1}{2^{4k}}\left\|[\nabla P_k,\psi]\frac{\big(\nabla^M\Phi_0\big)_Y}{\log t'}\right\|_{L^2}^2.$$

Using the bounds (6.15) and (6.19) in Lemma 6.3 for the last three terms, we obtain:

$$\|X\|_{L^2}^2 + \frac{1}{2^{2k}}\|\nabla Y\|_{L^2}^2 \lesssim \frac{1}{2^{2k}}\|\nabla P_k \nabla^M \mathcal{O}\|_{L^2}^2 + \int_0^t \frac{(t')^2}{2^{6k}}\left\|\nabla[P_k,\nabla_4]\frac{\big(\nabla^M\Phi_0\big)_Y}{\log t'}\right\|_{L^2}^2 +$$
$$+ \int_0^t \frac{(t')^2}{2^{6k}}\|[\nabla,\nabla_4]Y\|_{L^2}^2 + \int_0^t \frac{1}{2^{3k}}\left\|\frac{\big(\nabla^M\Phi_0\big)_Y}{\log t'}\right\|_{L^2}^2 + \int_0^t \frac{(t')^2}{2^{4k}}\left\|[\nabla_4,P_k]\nabla_t\frac{\big(\nabla^M\Phi_0\big)_Y}{\log t'}\right\|_{L^2}^2 +$$
$$+ \int_0^t \|Z\|_{L^2}^2 + \int_0^t \frac{1}{2^{2k}}\left\|\frac{\nabla\nabla_t R_k \nabla^M\mathcal{O}}{\log t'}\right\|_{L^2}^2.$$

Finally, using the estimates (6.31)-(6.34) in Lemma 6.8 for the last two terms, we get for $\underline{P}_k^2 = P_k$:

$$\|X\|_{L^2}^2 + \frac{1}{2^{2k}}\|\nabla Y\|_{L^2}^2 \lesssim \frac{1}{2^{2k}}\|\nabla P_k \nabla^M \mathcal{O}\|_{L^2}^2 + \frac{1}{2^{2k}}\|\underline{P}_k\nabla^M\mathcal{O}\|_{H^1}^2 + \int_0^t \frac{1}{2^{3k}}\left\|\frac{\big(\nabla^M\Phi_0\big)_Y}{\log t}\right\|_{L^2}^2$$
$$+ \int_0^t \frac{(t')^2}{2^{6k}}\left\|\nabla[P_k,\nabla_4]\frac{\big(\nabla^M\Phi_0\big)_Y}{\log t}\right\|_{L^2}^2 + \int_0^t \frac{(t')^2}{2^{4k}}\left\|[\nabla_4,P_k]\nabla_t\frac{\big(\nabla^M\Phi_0\big)_Y}{\log t}\right\|_{L^2}^2.$$

Changing coordinates back to τ we obtain the desired estimate. $\square$

The right hand side of the estimate in Proposition 7.8 cannot be bounded using the top order terms on the left hand side. However, as explained in Section 1.3.3 of the introduction, we can bound these terms using the lower order estimates (including the fractional estimates) and the commutator estimate in Section 7.2. We obtain our main low frequency regime estimates in Proposition 7.9.

Proposition 7.9. *For any $k \geq 0$ and $\tau \leq 2^{-k-1}$, we have that $\big(\nabla^M\Phi_0\big)_Y$ satisfies the estimate:*

$$\left\|P_k\nabla_\tau\frac{\big(\nabla^M\Phi_0\big)_Y}{\log 2^k\tau} - R_k\nabla^M\mathcal{O}\partial_\tau\Big(\frac{1}{\log 2^k\tau}\Big)\right\|_{L^2}^2 + \left\|\nabla\Big(P_k\frac{\big(\nabla^M\Phi_0\big)_Y}{\log 2^k\tau} - \frac{R_k\nabla^M\mathcal{O}}{\log 2^k\tau}\Big)\right\|_{L^2}^2 \lesssim \tag{7.22}$$
$$\lesssim \|\nabla P_k\nabla^M\mathcal{O}\|_{L^2}^2 + \|\underline{P}_k\nabla^M\mathcal{O}\|_{H^1}^2 + 2^{-k/2}\|\mathcal{O}\|_{H^{M+1/2+\eta}}^2 + 2^{-2k}\int_0^\tau \|\tau'\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\|_{L^2}^2.$$

Moreover, we also have the estimate:

$$2^{2k}\left\|P_k\frac{\big(\nabla^M\Phi_0\big)_Y}{\log 2^k\tau} - \frac{R_k\nabla^M\mathcal{O}}{\log 2^k\tau}\right\|_{L^2}^2 \lesssim \|\nabla P_k\nabla^M\mathcal{O}\|_{L^2}^2 + \|\underline{P}_k\nabla^M\mathcal{O}\|_{H^1}^2 + \tag{7.23}$$
$$+ 2^{-k/2}\|\mathcal{O}\|_{H^{M+1/2+\eta}}^2 + 2^{-2k}\int_0^\tau \|\tau'\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\|_{L^2}^2.$$

Proof. We start with the proof of the estimate (7.22). We recall the notation $\mathcal{C} = \big(\nabla^M\Phi_0\big)_Y - \nabla\big(\nabla^{M-1}\Phi_0\big)_Y$ in Section 7.2. We can rewrite the estimate in Proposition 7.8 as:

$$\Big\|P_k\nabla_\tau\frac{\big(\nabla^M\Phi_0\big)_Y}{\log 2^k\tau} - R_k\nabla^M\mathcal{O}\partial_\tau\Big(\frac{1}{\log 2^k\tau}\Big)\Big\|_{L^2}^2 + \Big\|\nabla\Big(P_k\frac{\big(\nabla^M\Phi_0\big)_Y}{\log 2^k\tau} - \frac{R_k\nabla^M\mathcal{O}}{\log 2^k\tau}\Big)\Big\|_{L^2}^2 \lesssim$$

$$\lesssim \big\|\nabla P_k\nabla^M\mathcal{O}\big\|_{L^2}^2 + \big\|\underline{P}_k\nabla^M\mathcal{O}\big\|_{H^1}^2 + \int_0^\tau\frac{(\tau')^2}{2^k}\Big\|\nabla[P_k,\nabla_4]\frac{\mathcal{C}}{\log 2^k\tau'}\Big\|_{L^2}^2 +$$

$$+\int_0^\tau\frac{(\tau')^2}{2^k}\Big\|[P_k,\nabla_4]\nabla_\tau\frac{\mathcal{C}}{\log 2^k\tau'}\Big\|_{L^2}^2 + \int_0^\tau\Big\|\frac{\mathcal{C}}{\log 2^k\tau'}\Big\|_{L^2}^2 +$$

$$+\int_0^\tau\frac{(\tau')^2}{2^k}\Big\|[P_k,\nabla_4]\frac{\nabla\big(\nabla^{M-1}\Phi_0\big)_Y}{\log 2^k\tau'}\Big\|_{H^1}^2 + \int_0^\tau\frac{(\tau')^2}{2^k}\Big\|[P_k,\nabla_4]\nabla\frac{\nabla_\tau\big(\nabla^{M-1}\Phi_0\big)_Y}{\log 2^k\tau'}\Big\|_{L^2}^2 +$$

$$+\int_0^\tau\frac{(\tau')^2}{2^k}\Big\|[P_k,\nabla_4]\frac{[\nabla,\nabla_\tau]\big(\nabla^{M-1}\Phi_0\big)_Y}{\log 2^k\tau'}\Big\|_{L^2}^2 + \int_0^\tau\Big\|\frac{\nabla\big(\nabla^{M-1}\Phi_0\big)_Y}{\log 2^k\tau'}\Big\|_{L^2}^2.$$

We use the estimates (6.17)-(6.18) in Lemma 6.3 for the terms with $\mathcal{C}$ and Lemma 6.5 for the terms with $\big(\nabla^{M-1}\Phi_0\big)_Y$:

$$\Big\|P_k\nabla_\tau\frac{\big(\nabla^M\Phi_0\big)_Y}{\log 2^k\tau} - R_k\nabla^M\mathcal{O}\partial_\tau\Big(\frac{1}{\log 2^k\tau}\Big)\Big\|_{L^2}^2 + \Big\|\nabla\Big(P_k\frac{\big(\nabla^M\Phi_0\big)_Y}{\log 2^k\tau} - \frac{R_k\nabla^M\mathcal{O}}{\log 2^k\tau}\Big)\Big\|_{L^2}^2 \lesssim$$

$$\lesssim \big\|\nabla P_k\nabla^M\mathcal{O}\big\|_{L^2}^2 + \big\|\underline{P}_k\nabla^M\mathcal{O}\big\|_{H^1}^2 + \int_0^\tau(\tau')^2\Big\|\frac{\nabla\big(\nabla^{M-1}\Phi_0\big)_Y}{\log 2^k\tau'}\Big\|_{H^{1/2}}^2 +$$

$$+\int_0^\tau(\tau')^2\Big\|\frac{\nabla_\tau\big(\nabla^{M-1}\Phi_0\big)_Y}{\log 2^k\tau'}\Big\|_{H^{1/2}}^2 + \int_0^\tau\Big\|\frac{\big(\nabla^{M-1}\Phi_0\big)_Y}{\log 2^k\tau'}\Big\|_{H^1}^2 +$$

$$+\int_0^\tau\frac{(\tau')^2}{2^k}\Big\|\nabla\frac{\mathcal{C}}{\log 2^k\tau'}\Big\|_{L^2}^2 + \int_0^\tau\frac{(\tau')^2}{2^k}\Big\|\frac{\nabla_\tau\mathcal{C}}{\log 2^k\tau'}\Big\|_{L^2}^2 + \int_0^\tau\Big\|\frac{\mathcal{C}}{\log 2^k\tau'}\Big\|_{L^2}^2.$$

We then use the lower order estimates in Proposition 7.2 to get:

$$\Big\|P_k\nabla_\tau\frac{\big(\nabla^M\Phi_0\big)_Y}{\log 2^k\tau} - R_k\nabla^M\mathcal{O}\partial_\tau\Big(\frac{1}{\log 2^k\tau}\Big)\Big\|_{L^2}^2 + \Big\|\nabla\Big(P_k\frac{\big(\nabla^M\Phi_0\big)_Y}{\log 2^k\tau} - \frac{R_k\nabla^M\mathcal{O}}{\log 2^k\tau}\Big)\Big\|_{L^2}^2 \lesssim$$

$$\lesssim \big\|\nabla P_k\nabla^M\mathcal{O}\big\|_{L^2}^2 + \big\|\underline{P}_k\nabla^M\mathcal{O}\big\|_{H^1}^2 + 2^{-k/2}\|\mathcal{O}\|_{H^{M+1/2+\eta}}^2 +$$

$$+\int_0^\tau\frac{(\tau')^2}{2^k}\Big\|\nabla\frac{\mathcal{C}}{\log 2^k\tau'}\Big\|_{L^2}^2 + \int_0^\tau\frac{(\tau')^2}{2^k}\Big\|\frac{\nabla_\tau\mathcal{C}}{\log 2^k\tau'}\Big\|_{L^2}^2 + \int_0^\tau\Big\|\frac{\mathcal{C}}{\log 2^k\tau'}\Big\|_{L^2}^2.$$

Finally, we use the commutator estimates (7.12)-(7.13) to conclude the proof of (7.22).

Next, we prove the estimate (7.23). Using (7.16) and (7.14), we have that:

$$\Big\|P_k\frac{\big(\nabla^M\Phi_0\big)_Y}{\log 2^k\tau} - \frac{R_k\nabla^M\mathcal{O}}{\log 2^k\tau}\Big\|_{L^2}^2 \lesssim \big\|P_k\nabla^M\mathcal{O}\big\|_{L^2}^2 + 2^{-k}\int_0^\tau\Big\|\nabla_\tau\Big(P_k\frac{\big(\nabla^M\Phi_0\big)_Y}{\log 2^k\tau'} - \frac{R_k\nabla^M\mathcal{O}}{\log 2^k\tau'}\Big)\Big\|_{L^2}^2$$

$$+ 2^k \int_0^{\tau} \left\| P_k \frac{\big(\nabla^M \Phi_0\big)_Y}{\log 2^k\tau'} - \frac{R_k \nabla^M \mathcal{O}}{\log 2^k\tau'} \right\|^2_{L^2}.$$

We get by the already established estimate (7.22) and Gronwall:

$$2^{2k} \left\| P_k \frac{\big(\nabla^M \Phi_0\big)_Y}{\log 2^k\tau} - \frac{R_k \nabla^M \mathcal{O}}{\log 2^k\tau} \right\|^2_{L^2} \lesssim$$

$$\lesssim 2^{2k} \big\| P_k \nabla^M \mathcal{O} \big\|^2_{L^2} + \big\| \nabla P_k \nabla^M \mathcal{O} \big\|^2_{L^2} + \big\| \underline{P}_k \nabla^M \mathcal{O} \big\|^2_{H^1} + 2^{-k/2} \big\| \mathcal{O} \big\|^2_{H^{M+1/2+\eta}}$$

$$+ 2^{-2k} \int_0^{\tau} \big\| \tau' \nabla_\tau \big(\nabla^M \Phi_0\big)_Y \big\|^2_{L^2} + 2^{-k} \int_0^{\tau} \left\| \frac{\mathcal{C}}{\log 2^k\tau} \right\|^2_{L^2} + 2^{-k} \int_0^{\tau} \left\| \frac{\big(\nabla^{M-1} \Phi_0\big)_Y}{\log 2^k\tau} \right\|^2_{H^1}.$$

We conclude the proof of (7.23) using the lower order estimate (7.11) and the commutator estimate (7.13). □

We conclude this subsection by using the above results in order to obtain bounds at $\tau = 2^{-k-1}$. These will serve as estimates on the initial data in the high frequency regime $\tau \in [2^{-k-1}, 1]$ in Section 7.3.2.

Corollary 7.2. *For any $k \geq 0$ and $\tau = 2^{-k-1}$ we have the estimate:*

$$\Big(\big\| P_k \nabla_\tau \big(\nabla^M \Phi_0\big)_Y \big\|^2_{L^2} + 2^{2k} \big\| P_k \big(\nabla^M \Phi_0\big)_Y \big\|^2_{L^2} + \big\| \nabla P_k \big(\nabla^M \Phi_0\big)_Y \big\|^2_{L^2} \Big) \Big|_{\tau = 2^{-k-1}} \lesssim$$

$$\lesssim \big\| \nabla P_k \nabla^M \mathcal{O} \big\|^2_{L^2} + \big\| \underline{P}_k \nabla^M \mathcal{O} \big\|^2_{H^1} + 2^{-k/2} \big\| \mathcal{O} \big\|^2_{H^{M+1}} + 2^{-2k} \int_0^{2^{-k-1}} \big\| \tau' \nabla_\tau \big(\nabla^M \Phi_0\big)_Y \big\|^2_{L^2}.$$

Proof. The low frequency estimates in Proposition 7.9 imply for $\tau = 2^{-k-1}$:

$$\Big(\big\| P_k \nabla_\tau \big(\nabla^M \Phi_0\big)_Y \big\|^2_{L^2} + 2^{2k} \big\| P_k \big(\nabla^M \Phi_0\big)_Y \big\|^2_{L^2} + \big\| \nabla P_k \big(\nabla^M \Phi_0\big)_Y \big\|^2_{L^2} \Big) \Big|_{\tau = 2^{-k-1}} \lesssim$$

$$\lesssim \big\| \nabla P_k \nabla^M \mathcal{O} \big\|^2_{L^2} + \big\| \underline{P}_k \nabla^M \mathcal{O} \big\|^2_{H^1} + 2^{-k/2} \big\| \mathcal{O} \big\|^2_{H^{M+1/2+\eta}} +$$

$$+ 2^{2k} \big\| R_k \nabla^M \mathcal{O} \big\|^2_{L^2} + \big\| \nabla R_k \nabla^M \mathcal{O} \big\|^2_{L^2} + 2^{-2k} \int_0^{2^{-k-1}} \big\| \tau' \nabla_\tau \big(\nabla^M \Phi_0\big)_Y \big\|^2_{L^2}.$$

We conclude using estimates (6.32)-(6.33) in Lemma 6.8. □

7.3.2 High Frequency Regime Estimates

In this section, we prove a high frequency regime estimate for the singular component $\big(\nabla^M \Phi_0\big)_Y$. As in the case of [Cic23], according to the Bessel function type asymptotics, we prove an energy estimate for $\sqrt{2^k\tau} \cdot P_k \big(\nabla^M \Phi_0\big)_Y$, which implies the 1/2 gain of regularity at $\tau = 1$ compared to $\tau = 2^{-k-1.}$

Proposition 7.10. *For any $k \geq 0,\ \tau \in \left[2^{-k-1}, 1\right]$ we have the estimate for the singular component:*

$$\begin{aligned}
&\tau \left\| P_k \nabla_\tau \left(\nabla^M \Phi_0\right)_Y \right\|_{L^2}^2 + \frac{1}{\tau} \left\| P_k \left(\nabla^M \Phi_0\right)_Y \right\|_{L^2}^2 + \tau \left\| \nabla P_k \left(\nabla^M \Phi_0\right)_Y \right\|_{L^2}^2 \lesssim \\
&\lesssim \frac{1}{2^k} \left(\left\| P_k \nabla_\tau \left(\nabla^M \Phi_0\right)_Y \right\|_{L^2}^2 + 2^{2k} \left\| P_k \left(\nabla^M \Phi_0\right)_Y \right\|_{L^2}^2 + \left\| \nabla P_k \left(\nabla^M \Phi_0\right)_Y \right\|_{L^2}^2 \right) \Big|_{\tau = 2^{-k-1}} \\
&+ \int_{2^{-k-1}}^{\tau} \tau' \left\| \underline{\widetilde{P}}_k \left(\nabla^M \Phi_0\right)_Y \right\|_{L^2}^2 + \int_{2^{-k-1}}^{\tau} (\tau')^3 \left\| \underline{\widetilde{P}}_k \nabla \left(\nabla^M \Phi_0\right)_Y \right\|_{L^2}^2 + \\
&+ \int_{2^{-k-1}}^{\tau} (\tau')^3 \left\| \underline{\widetilde{P}}_k \nabla_\tau \left(\nabla^M \Phi_0\right)_Y \right\|_{L^2}^2 + \int_{2^{-k-1}}^{\tau} \frac{\tau'}{2^{2k}} \left\| \left(\nabla^M \Phi_0\right)_Y \right\|_{H^1}^2 + \int_{2^{-k-1}}^{\tau} \frac{(\tau')^3}{2^{2k}} \left\| \nabla_\tau \left(\nabla^M \Phi_0\right)_Y \right\|_{L^2}^2 .
\end{aligned}$$

Proof. We denote $\xi = \left(\nabla^M \Phi_0\right)_Y$ and we introduce the new time variable $t = 2^k \tau$. We rewrite (7.4):

$$\nabla_t \big(\nabla_t \xi\big) + \frac{1}{t} \nabla_t \xi - \frac{4}{2^{2k}} \Delta \xi = \frac{1}{2^{2k}} \psi \nabla \xi .$$

We multiply by $\sqrt{t}$ to get:

$$\nabla_t \big(\nabla_t (\xi \sqrt{t})\big) + \frac{1}{4t^2} \xi \sqrt{t} - \frac{4}{2^{2k}} \Delta \xi \sqrt{t} = \frac{1}{2^{2k}} \psi \nabla \xi \sqrt{t}. \tag{7.24}$$

For any $k \geq 0$, we apply P_k to (7.24) to obtain the equation:

$$\nabla_t \big(P_k \nabla_t (\xi \sqrt{t})\big) + \frac{1}{4t^2} P_k \xi \sqrt{t} - \frac{4}{2^{2k}} \Delta P_k \xi \sqrt{t} = \frac{P_k \big(\psi \nabla \xi \sqrt{t}\big)}{2^{2k}} + [\nabla_t, P_k] \nabla_t \xi \sqrt{t}. \tag{7.25}$$

We contract the equation (7.25) with $P_k \nabla_t (\xi \sqrt{t})$ and integrate by parts to obtain the energy estimate:

$$\begin{aligned}
&\left\| P_k \nabla_t \xi \sqrt{t} \right\|_{L^2}^2 + \frac{1}{t^2} \left\| P_k \xi \sqrt{t} \right\|_{L^2}^2 + \frac{1}{2^{2k}} \left\| \nabla P_k \xi \sqrt{t} \right\|_{L^2}^2 + \int_{1/2}^{t} \frac{1}{(t')^2} \left\| P_k \xi \right\|_{L^2}^2 \lesssim \\
&\lesssim \left(\left\| P_k \nabla_t \xi \right\|_{L^2}^2 + \left\| P_k \xi \right\|_{L^2}^2 + \frac{1}{2^{2k}} \left\| \nabla P_k \xi \right\|_{L^2}^2 \right) \Big|_{t = 1/2} + \int_{1/2}^{t} \int_{S^n} \frac{1}{(t')^2} \left| P_k \xi \sqrt{t} \right| \cdot \left| [P_k, \nabla_t] \xi \sqrt{t'} \right| \\
&+ \int_{1/2}^{t} \int_{S^n} \frac{1}{2^{2k}} \left| \nabla P_k \xi \sqrt{t'} \right| \cdot \left| \nabla [P_k, \nabla_t] \xi \sqrt{t'} \right| + \int_{1/2}^{t} \int_{S^n} \frac{1}{2^{2k}} \left| \nabla P_k \xi \sqrt{t'} \right| \cdot \left| [\nabla, \nabla_t] P_k \xi \sqrt{t'} \right| \\
&+ \int_{1/2}^{t} \int_{S^n} \frac{1}{2^{2k}} \left| P_k \nabla_t \xi \sqrt{t'} \right| \cdot \left| \nabla P_k \xi \sqrt{t'} \right| + \int_{1/2}^{t} \int_{S^n} \frac{1}{2^{2k}} \left| P_k \nabla_t \xi \sqrt{t'} \right| \cdot \left| [P_k, \nabla] \xi \sqrt{t'} \right| \\
&+ \int_{1/2}^{t} \int_{S^n} \frac{1}{2^{2k}} \left| P_k \nabla_t \xi \sqrt{t'} \right| \cdot \left| [P_k, \psi] \nabla \xi \sqrt{t'} \right| + \int_{1/2}^{t} \int_{S^n} \left| P_k \nabla_t \xi \sqrt{t'} \right| \cdot \left| [\nabla_t, P_k] \nabla_t \xi \sqrt{t'} \right| .
\end{aligned}$$

We point out that the good bulk term simplifies our analysis significantly, unlike the case of the second model system studied in Section 8. We use the bounds (6.22), (6.24) in Lemma 6.4 and

Gronwall for $t \in \left[1/2, 2^k\right]$ to get:

$$\begin{aligned}
&\left\|P_k \nabla_t \xi \sqrt{t}\right\|_{L^2}^2 + \frac{1}{t^2}\left\|P_k \xi \sqrt{t}\right\|_{L^2}^2 + \frac{1}{2^{2k}}\left\|\nabla P_k \xi \sqrt{t}\right\|_{L^2}^2 \lesssim \\
&\lesssim \left(\left\|P_k \nabla_t \xi\right\|_{L^2}^2 + \left\|P_k \xi\right\|_{L^2}^2 + \frac{1}{2^{2k}}\left\|\nabla P_k \xi\right\|_{L^2}^2\right)\Big|_{t=1/2} \\
&+ \int_{1/2}^{t} \frac{1}{2^{2k} t'}\left\|P_k \xi \sqrt{t'}\right\|_{L^2} \cdot \left(\left\|\underline{\widetilde{P}}_k \xi \sqrt{t'}\right\|_{L^2} + 2^{-k}\left\|\xi \sqrt{t'}\right\|_{L^2}\right) + \\
&+ \int_{1/2}^{t} \frac{1}{2^{2k}}\left\|P_k \nabla_t \xi \sqrt{t'}\right\|_{L^2} \cdot \left\|\nabla P_k \xi \sqrt{t'}\right\|_{L^2} + \\
&+ \int_{1/2}^{t} \frac{t'}{2^{4k}}\left\|\nabla P_k \xi \sqrt{t'}\right\|_{L^2} \cdot \left(\left\|\underline{\widetilde{P}}_k \nabla \xi \sqrt{t'}\right\|_{L^2} + 2^{-k}\left\|\xi \sqrt{t'}\right\|_{H^1}\right) + \\
&+ \int_{1/2}^{t} \frac{1}{2^{3k}}\left\|P_k \nabla_t \xi \sqrt{t'}\right\|_{L^2} \cdot \left\|\xi \sqrt{t'}\right\|_{L^2} + \int_{1/2}^{t} \frac{1}{2^{3k}}\left\|P_k \nabla_t \xi \sqrt{t'}\right\|_{L^2} \cdot \left\|\nabla \xi \sqrt{t'}\right\|_{L^2} + \\
&+ \int_{1/2}^{t} \frac{t'}{2^{2k}}\left\|P_k \nabla_t \xi \sqrt{t'}\right\|_{L^2}\left(\left\|\underline{\widetilde{P}}_k \nabla_t \xi \sqrt{t'}\right\|_{L^2} + 2^{-k}\left\|\nabla_t \xi \sqrt{t'}\right\|_{L^2}\right).
\end{aligned}$$

We use Gronwall again for $t \in \left[1/2, 2^k\right]$ to get the estimate:

$$\begin{aligned}
&t\left\|P_k \nabla_t \xi\right\|_{L^2}^2 + \frac{1}{t}\left\|P_k \xi\right\|_{L^2}^2 + \frac{t}{2^{2k}}\left\|\nabla P_k \xi\right\|_{L^2}^2 \lesssim \\
&\lesssim \left(\left\|P_k \nabla_t \xi\right\|_{L^2}^2 + \left\|P_k \xi\right\|_{L^2}^2 + \frac{1}{2^{2k}}\left\|\nabla P_k \xi\right\|_{L^2}^2\right)\Big|_{t=\frac{1}{2}} + \int_{1/2}^{t} \frac{1}{2^{3k}}\left\|\underline{\widetilde{P}}_k \xi \sqrt{t'}\right\|_{L^2}^2 \\
&+ \int_{1/2}^{t} \frac{(t')^2}{2^{5k}}\left\|\underline{\widetilde{P}}_k \nabla \xi \sqrt{t'}\right\|_{L^2}^2 + \int_{1/2}^{t} \frac{(t')^2}{2^{3k}}\left\|\underline{\widetilde{P}}_k \nabla_t \xi \sqrt{t'}\right\|_{L^2}^2 + \\
&+ \int_{1/2}^{t} \frac{1}{2^{5k}}\left\|\xi \sqrt{t'}\right\|_{H^1}^2 + \int_{1/2}^{t} \frac{(t')^2}{2^{5k}}\left\|\nabla_t \xi \sqrt{t'}\right\|_{L^2}^2.
\end{aligned}$$

We change variables to τ and we obtain the conclusion. $\square$

7.3.3 Main Result for the Singular Component

In this section we combine the low frequency regime and the high frequency regime estimates for the singular component $\left(\nabla^M \Phi_0\right)_Y$ in order to prove Theorem 7.2. As remarked in Section 1.3.3 of the introduction, due to presence of different projection operators in the estimates established above, we must sum the estimates obtained for each LP projection before being able to bound the error terms.

Proof of Theorem 7.2. Step 1. The improved high frequency regime estimate. The result in Proposition 7.10 implies that for $\tau \in \big[2^{-k-1},1\big]$:

$$2^k\tau\big\|P_k\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2+\frac{2^k}{\tau}\big\|P_k\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2+2^k\tau\big\|\nabla P_k\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2\lesssim$$

$$\lesssim\bigg(\big\|P_k\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2+2^{2k}\big\|P_k\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2+\big\|\nabla P_k\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2\bigg)\bigg|_{\tau=2^{-k-1}}$$

$$+2^k\int_{2^{-k-1}}^{\tau}\tau'\big\|\underline{\widetilde{P}}_k\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2+2^k\int_{2^{-k-1}}^{\tau}(\tau')^3\big\|\underline{\widetilde{P}}_k\nabla\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2+$$

$$+2^k\int_{2^{-k-1}}^{\tau}(\tau')^3\big\|\underline{\widetilde{P}}_k\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2$$

$$+\int_{2^{-k-1}}^{\tau}\frac{\tau'}{2^k}\big\|\big(\nabla^M\Phi_0\big)_Y\big\|_{H^1}^2+\int_{2^{-k-1}}^{\tau}\frac{(\tau')^3}{2^k}\big\|\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2.$$

Using the low frequency regime estimates (7.22)-(7.23) in Proposition 7.9 to bound the boundary term at $\tau=2^{-k-1}$, we get the improved high frequency regime estimate for $\tau\in\big[2^{-k-1},1\big]$:

$$2^k\tau\big\|P_k\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2+\frac{2^k}{\tau}\big\|P_k\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2+2^k\tau\big\|\nabla P_k\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2\lesssim$$

$$\lesssim\big\|\nabla P_k\nabla^M\mathcal{O}\big\|_{L^2}^2+\big\|\underline{P}_k\nabla^M\mathcal{O}\big\|_{H^1}^2+2^{-k/2}\big\|\mathcal{O}\big\|_{H^{M+1}}^2+2^{-2k}\int_0^{2^{-k-1}}\big\|\tau'\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2+$$

$$+2^k\int_{2^{-k-1}}^{\tau}\tau'\big\|\underline{\widetilde{P}}_k\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2+2^k\int_{2^{-k-1}}^{\tau}(\tau')^3\big\|\underline{\widetilde{P}}_k\nabla\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2+$$

$$+2^k\int_{2^{-k-1}}^{\tau}(\tau')^3\big\|\underline{\widetilde{P}}_k\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2$$

$$+\int_{2^{-k-1}}^{\tau}\frac{\tau'}{2^k}\big\|\big(\nabla^M\Phi_0\big)_Y\big\|_{H^1}^2+\int_{2^{-k-1}}^{\tau}\frac{(\tau')^3}{2^k}\big\|\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2.$$

Step 2. Bounding the non-negative frequencies. We define the following energy for all $k\geq 0$:

$$2^kE_k^2(\tau)=2^k\tau^2\big\|P_k\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2+2^k\tau\big\|P_k\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2+$$

$$+2^k\tau^2\big\|\nabla P_k\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2+\tau\big\|\nabla P_k\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2.$$

For any $\tau\in(0,1]$ we can write:

$$\sum_{k\geq 0}2^kE_k^2(\tau)=\sum_{\tau\leq 2^{-k-1}}2^kE_k^2(\tau)+\sum_{\tau>2^{-k-1}}2^kE_k^2(\tau). \tag{7.26}$$

We use our previous estimates to bound (7.26). The improved high frequency regime estimate implies the bound:

$$\sum_{\tau>2^{-k-1}} 2^k E_k^2 \lesssim \big\|\mathcal{O}\big\|^2_{H^{M+1}} + \int_0^\tau (\tau')^2 \big\|\nabla_\tau \big(\nabla^M \Phi_0\big)_Y\big\|^2_{H^{1/2}} + \int_0^\tau (\tau')^2 \big\|\nabla \big(\nabla^M \Phi_0\big)_Y\big\|^2_{H^{1/2}} +$$

$$+ \int_0^\tau \tau' \big\|\nabla \big(\nabla^M \Phi_0\big)_Y\big\|^2_{L^2} + \int_0^\tau \tau' \big\|\big(\nabla^M \Phi_0\big)_Y\big\|^2_{H^{1/2}}.$$

The low frequency regime estimates (7.22)-(7.23) in Proposition 7.9 imply:

$$\sum_{\tau\leq 2^{-k-1}} 2^k \tau^2 \big\|\nabla P_k \big(\nabla^M \Phi_0\big)_Y\big\|^2_{L^2} \lesssim \sum_{\tau\leq 2^{-k-1}} \tau \big\|\nabla P_k \big(\nabla^M \Phi_0\big)_Y\big\|^2_{L^2} \lesssim$$

$$\lesssim \sum_{\tau\leq 2^{-k-1}} \tau \big\|\nabla R_k \nabla^M \mathcal{O}\big\|^2_{L^2} + \sum_{\tau\leq 2^{-k-1}} \left\|\nabla\left(P_k \frac{\big(\nabla^M \Phi_0\big)_Y}{\log 2^k\tau} - \frac{R_k \nabla^M \mathcal{O}}{\log 2^k\tau}\right)\right\|^2_{L^2}$$

$$\lesssim \big\|\mathcal{O}\big\|^2_{H^{M+1}} + \int_0^\tau \big\|\tau' \nabla_\tau \big(\nabla^M \Phi_0\big)_Y\big\|^2_{L^2}.$$

Similarly, we also have the bound:

$$\sum_{\tau\leq 2^{-k-1}} 2^k \tau^2 \big\|P_k \nabla_\tau \big(\nabla^M \Phi_0\big)_Y\big\|^2_{L^2} \lesssim$$

$$\lesssim \sum_{\tau\leq 2^{-k-1}} 2^k \tau^2 \big|\log(2^k\tau)\big|^2 \cdot \left\|P_k \nabla_\tau \frac{\big(\nabla^M \Phi_0\big)_Y}{\log 2^k\tau}\right\|^2_{L^2} + \sum_{\tau\leq 2^{-k-1}} 2^k \left\|\frac{P_k \big(\nabla^M \Phi_0\big)_Y}{\log 2^k\tau}\right\|^2_{L^2}$$

$$\lesssim \sum_{\tau\leq 2^{-k-1}} 2^k \tau^2 \big|\log(2^k\tau)\big|^2 \cdot \left\|P_k \nabla_\tau \frac{\big(\nabla^M \Phi_0\big)_Y}{\log 2^k\tau} - R_k \nabla^M \mathcal{O} \partial_\tau \left(\frac{1}{\log 2^k\tau}\right)\right\|^2_{L^2} +$$

$$+ \sum_{\tau\leq 2^{-k-1}} 2^k \big\|R_k \nabla^M \mathcal{O}\big\|^2_{L^2} + \sum_{\tau\leq 2^{-k-1}} 2^k \left\|\frac{P_k \big(\nabla^M \Phi_0\big)_Y}{\log 2^k\tau}\right\|^2_{L^2}$$

$$\lesssim \big\|\mathcal{O}\big\|^2_{H^{M+1}} + \int_0^\tau \big\|\tau' \nabla_\tau \big(\nabla^M \Phi_0\big)_Y\big\|^2_{L^2}.$$

Next, we have the bound:

$$\sum_{\tau\leq 2^{-k-1}} 2^k \tau \big\|P_k \big(\nabla^M \Phi_0\big)_Y\big\|^2_{L^2} \lesssim$$

$$\lesssim \sum_{\tau\leq 2^{-k-1}} 2^{2k} \left\|P_k \frac{\big(\nabla^M \Phi_0\big)_Y}{\log 2^k\tau} - \frac{R_k \nabla^M \mathcal{O}}{\log 2^k\tau}\right\|^2_{L^2} + \sum_{\tau\leq 2^{-k-1}} 2^k \big\|R_k \nabla^M \mathcal{O}\big\|^2_{L^2}$$

$$\lesssim \big\|\mathcal{O}\big\|^2_{H^{M+1}} + \int_0^\tau \big\|\tau' \nabla_\tau \big(\nabla^M \Phi_0\big)_Y\big\|^2_{L^2}.$$

As a result, we obtain the following bound for the sum in (7.26):

$$\sum_{k\geq 0} 2^k E_k^2 \lesssim \big\|\mathcal{O}\big\|^2_{H^{M+1}} + \int_0^\tau (\tau')^2 \big\|\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|^2_{H^{1/2}} + \tag{7.27}$$

$$+\int_0^\tau (\tau')^2 \big\|\nabla\big(\nabla^M\Phi_0\big)_Y\big\|^2_{H^{1/2}} + \int_0^\tau \tau' \big\|\nabla\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2} + \int_0^\tau \tau' \big\|\big(\nabla^M\Phi_0\big)_Y\big\|^2_{H^{1/2}}.$$

Step 3. Bounding the negative frequencies. In order to prove (7.7), we also need to deal with the negative frequencies. According to [KR06, Theorem 5.5], for any $k<0$ we also have $\|P_k\nabla F\|_{L^2} \lesssim 2^k\|F\|_{L^2}$. Thus, we have:

$$\sum_{k<0} 2^k\tau^2\big\|P_k\nabla\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2} + \tau\big\|P_k\nabla\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2} \lesssim \sum_{k<0} 2^k\tau\big\|\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2}$$

$$\lesssim \tau\big\|\mathcal{C}\big\|^2_{L^2} + \tau\big\|\nabla\big(\nabla^{M-1}\Phi_0\big)_Y\big\|^2_{L^2},$$

$$\sum_{k<0} 2^k\tau\big\|P_k\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2} \lesssim \tau\big\|\mathcal{C}\big\|^2_{L^2} + \tau\big\|\nabla\big(\nabla^{M-1}\Phi_0\big)_Y\big\|^2_{L^2},$$

$$\sum_{k<0} 2^k\tau^2\big\|P_k\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2} \lesssim$$

$$\lesssim \tau^2\big\|\nabla_\tau\mathcal{C}\big\|^2_{L^2} + \sum_{k<0} 2^k\tau^2\big\|[\nabla,\nabla_\tau]\big(\nabla^{M-1}\Phi_0\big)_Y\big\|^2_{L^2} + \sum_{k<0} 2^{3k}\tau^2\big\|\nabla_\tau\big(\nabla^{M-1}\Phi_0\big)_Y\big\|^2_{L^2}.$$

We proved that for the negative frequencies we have:

$$\sum_{k<0} 2^k\tau^2\big\|P_k\nabla\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2} + \tau\big\|P_k\nabla\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2} +$$

$$+2^k\tau\big\|P_k\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2} + 2^k\tau^2\big\|P_k\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2}$$

$$\lesssim \tau^2\big\|\nabla_\tau\mathcal{C}\big\|^2_{L^2} + \tau\big\|\mathcal{C}\big\|^2_{L^2} + \tau\big\|\big(\nabla^{M-1}\Phi_0\big)_Y\big\|^2_{H^1} + \tau^2\big\|\nabla_\tau\big(\nabla^{M-1}\Phi_0\big)_Y\big\|^2_{L^2}$$

$$\lesssim \big\|\mathcal{O}\big\|^2_{H^{M+1}} + \int_0^\tau \big\|\tau'\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2}.$$

Step 4. The proof of (7.7) and (7.8). We obtain that:

$$\tau^2\big\|\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|^2_{H^{1/2}} + \tau\big\|\big(\nabla^M\Phi_0\big)_Y\big\|^2_{H^{1/2}} + \tau^2\big\|\nabla\big(\nabla^M\Phi_0\big)_Y\big\|^2_{H^{1/2}} + \tau\big\|\nabla\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2} \lesssim$$

$$\lesssim \tau\big\|\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2} + \sum_{k\geq 0} 2^k E_k^2 + \big\|\mathcal{O}\big\|^2_{H^{M+1}} + \int_0^\tau \big\|\tau'\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2}$$

$$\lesssim \big\|\mathcal{O}\big\|^2_{H^{M+1}} + \int_0^\tau (\tau')^2\big\|\nabla_\tau\big(\nabla^M\Phi_0\big)_Y\big\|^2_{H^{1/2}} + (\tau')^2\big\|\nabla\big(\nabla^M\Phi_0\big)_Y\big\|^2_{H^{1/2}} +$$

$$+\tau'\big\|\nabla\big(\nabla^M\Phi_0\big)_Y\big\|^2_{L^2} + \tau'\big\|\big(\nabla^M\Phi_0\big)_Y\big\|^2_{H^{1/2}}.$$

By Gronwall, we obtain (7.7). Next, we notice that for any $k \geq 0$ using the high frequency estimate we get:

$$\sum_{\tau > 2^{-k-1}} \big\|P_k \nabla \big(\nabla^M \Phi_0\big)_Y\big\|_{L^2}^2 \lesssim \big\|\mathcal{C}\big\|_{L^2}^2 + \big\|\nabla \big(\nabla^{M-1} \Phi_0\big)_Y\big\|_{L^2}^2 + \sum_{\tau > 2^{-k-1}} 2^k \tau \big\|\nabla P_k \big(\nabla^M \Phi_0\big)_Y\big\|_{L^2}^2 \lesssim$$

$$\lesssim \big(1 + |\log \tau|^2\big) \big\|\mathcal{O}\big\|_{H^{M+1}}^2 + \int_0^\tau (\tau')^2 \big\|\nabla_\tau \big(\nabla^M \Phi_0\big)_Y\big\|_{H^{1/2}}^2 + \int_0^\tau (\tau')^2 \big\|\nabla \big(\nabla^M \Phi_0\big)_Y\big\|_{H^{1/2}}^2 +$$

$$+ \int_0^\tau \tau' \big\|\nabla \big(\nabla^M \Phi_0\big)_Y\big\|_{L^2}^2 + \int_0^\tau \tau' \big\|\big(\nabla^M \Phi_0\big)_Y\big\|_{H^{1/2}}^2 \lesssim \big(1 + |\log \tau|^2\big) \big\|\mathcal{O}\big\|_{H^{M+1}}^2.$$

As before, we also have for any $k \geq 0$:

$$\sum_{\tau \leq 2^{-k-1}} \big\|\nabla P_k \big(\nabla^M \Phi_0\big)_Y\big\|_{L^2}^2 \lesssim$$

$$\lesssim \sum_{\tau \leq 2^{-k-1}} \big\|\nabla R_k \nabla^M \mathcal{O}\big\|_{L^2}^2 + \sum_{\tau \leq 2^{-k-1}} |\log \tau|^2 \bigg\|\nabla \bigg(P_k \frac{\big(\nabla^M \Phi_0\big)_Y}{\log 2^k \tau} - \frac{R_k \nabla^M \mathcal{O}}{\log 2^k \tau}\bigg)\bigg\|_{L^2}^2$$

$$\lesssim \big(1 + |\log \tau|^2\big) \big\|\mathcal{O}\big\|_{H^{M+1}}^2.$$

We get the same bound for $\sum_{\tau \leq 2^{-k-1}} \big\|P_k \nabla \big(\nabla^M \Phi_0\big)_Y\big\|_{L^2}^2$ by commutation. For the negative frequencies we have:

$$\sum_{k<0} \big\|P_k \nabla \big(\nabla^M \Phi_0\big)_Y\big\|_{L^2}^2 \lesssim \big\|\big(\nabla^M \Phi_0\big)_Y\big\|_{L^2}^2 \lesssim$$

$$\lesssim \big\|\mathcal{C}\big\|_{L^2}^2 + \big\|\nabla \big(\nabla^{M-1} \Phi_0\big)_Y\big\|_{L^2}^2 \lesssim \big(1 + |\log \tau|^2\big) \big\|\mathcal{O}\big\|_{H^{M+1}}^2.$$

This completes the proof of (7.8). ☐

7.4 Top order Estimates for the Regular Components

We prove top order estimates for the regular quantities $\big(\nabla^M \Phi_0\big)_J, \nabla^M \Phi_1, \dots \nabla^M \Phi_I$ as outlined in Section 1.3.3. Similarly to the case of the singular component $\big(\nabla^M \Phi_0\big)_Y$, the analysis requires a precise understanding of the behavior of the P_k projections of each component. We treat separately the low frequency regime $\tau \in (0, 2^{-k-1}]$ in Section 7.4.1 and the high frequency regime $\tau \in [2^{-k-1}, 1]$ in Section 7.4.2. Finally, we combine the estimates in Section 7.4.3 to complete the proof of Theorem 7.1.

7.4.1 Low Frequency Regime Estimates

We prove a low frequency regime estimate for the regular components for each $k \geq 0$ $P_k\big(\nabla^M\Phi_0\big)_J, P_k\nabla^M\Phi_1, \ldots, P_k\nabla^M\Phi_I$. The main idea is to propagate for $\tau \in (0, 2^{-k-1}]$ the L^2 bounds satisfied by the asymptotic data at $\mathcal{I}^-$, using ∇_τ as a multiplier.

Proposition 7.11. *For any $k \geq 0$ and $\tau \leq 2^{-k-1}$, we have the estimate:*

$$\begin{aligned}
&\big\|P_k\nabla_\tau\big(\nabla^M\Phi_0\big)_J\big\|^2_{L^2} + \big\|\nabla P_k\big(\nabla^M\Phi_0\big)_J\big\|^2_{L^2} + 2^{2k}\big\|P_k\big(\nabla^M\Phi_0\big)_J\big\|^2_{L^2} + \\
&+\sum_{i=1}^{I}\Big(\big\|P_k\nabla_\tau\nabla^M\Phi_i\big\|^2_{L^2} + \big\|\nabla P_k\nabla^M\Phi_i\big\|^2_{L^2} + 2^{2k}\big\|P_k\nabla^M\Phi_i\big\|^2_{L^2}\Big) \lesssim \\
&\lesssim \big\|\nabla P_k\mathfrak{h}_M\big\|^2_{L^2} + 2^{2k}\big\|P_k\mathfrak{h}_M\big\|^2_{L^2} + \sum_{i=1}^{I}\big\|\nabla P_k\nabla^M\Phi_i^0\big\|^2_{L^2} + \sum_{i=1}^{I}2^{2k}\big\|P_k\nabla^M\Phi_i^0\big\|^2_{L^2} + \\
&+\sum_{i=1}^{I}\int_0^\tau\Big(\frac{1}{2^{3k}}\big\|\nabla^M\Phi_i\big\|^2_{H^1} + \frac{(\tau')^2}{2^k}\big\|\nabla_\tau\nabla^M\Phi_i\big\|^2_{L^2}\Big)d\tau' + \int_0^\tau\frac{(\tau')^2}{2^k}\big\|\nabla_\tau\big(\nabla^M\Phi_0\big)_J\big\|^2_{L^2}d\tau' + \\
&+\int_0^\tau\frac{1}{2^{3k}}\big\|\big(\nabla^M\Phi_0\big)_J\big\|^2_{H^1}d\tau' + \sum_{i=0}^{I}\int_0^\tau\frac{1}{2^k}\big\|P_kF_M^i\big\|^2_{L^2}d\tau' + \int_0^\tau\frac{1}{2^k}\big\|P_k\big(\psi\nabla\big(\nabla^M\Phi_0\big)_Y\big)\big\|^2_{L^2}d\tau'.
\end{aligned}$$

Proof. We denote $\xi_0 = \big(\nabla^M\Phi_0\big)_J$ and for $1 \leq i \leq I$ we denote $\xi_i = \nabla^M\Phi_i$. We introduce the new time variable $t = 2^k\tau$. Equations (5.13) and (7.5) with $m = M$ can be written as follows for all $0 \leq i \leq I$:

$$\nabla_t\big(\nabla_t\xi_i\big) + \frac{1}{t}\nabla_t\xi_i - \frac{4}{2^{2k}}\cdot\Delta\xi_i = \sum_{j=0}^{I}\frac{1}{2^{2k}}\cdot\psi\nabla\xi_j + \frac{1}{2^{2k}}\cdot F_M^i + \frac{1}{2^{2k}}\cdot\psi\nabla\big(\nabla^M\Phi_0\big)_Y.$$

For any $k \geq 0$, we apply P_k to each equation:

$$\begin{aligned}
\nabla_t\big(P_k\nabla_t\xi_i\big) + \frac{1}{t}P_k\nabla_t\xi_i - \frac{4}{2^{2k}}\cdot\Delta P_k\xi_i = \sum_{j=0}^{I}\frac{1}{2^{2k}}\cdot\psi\nabla P_k\xi_j + \sum_{j=0}^{I}\frac{1}{2^{2k}}\psi[P_k,\nabla]\xi_j + \\
+\sum_{j=0}^{I}\frac{1}{2^{2k}}[P_k,\psi]\nabla\xi_j + \frac{1}{2^{2k}}\cdot P_kF_M^i + [\nabla_t, P_k]\nabla_t\xi_i + \frac{1}{2^{2k}}\cdot P_k\big(\psi\nabla\big(\nabla^M\Phi_0\big)_Y\big).
\end{aligned}$$

We contract each equation with $P_k\nabla_t\xi_i$ and integrate by parts to obtain the energy estimate:

$$\sum_{i=0}^{I}\big\|P_k\nabla_t\xi_i\big\|^2_{L^2} + \sum_{i=0}^{I}\frac{1}{2^{2k}}\big\|\nabla P_k\xi_i\big\|^2_{L^2} \lesssim$$

$$\lesssim \sum_{i=0}^{I}\frac{1}{2^{2k}}\big\|\nabla P_k\xi_i^0\big\|^2_{L^2} + \sum_{i=0}^{I}\int_0^t\frac{1}{2^{2k}}\big\|\nabla P_k\xi_i\big\|_{L^2}\cdot\big\|\nabla[P_k,\nabla_t]\xi_i\big\|_{L^2} +$$

$$+\sum_{i=0}^{I}\int_0^t \frac{1}{2^{2k}}\big\|\nabla P_k\xi_i\big\|_{L^2}\cdot\big\|[\nabla,\nabla_t]P_k\xi_i\big\|_{L^2}+\sum_{i=0}^{I}\int_0^t \frac{1}{2^{2k}}\big\|\nabla P_k\xi_i\big\|_{L^2}\cdot\big\|P_k\nabla_t\xi_i\big\|_{L^2}+$$

$$+\sum_{i=0}^{I}\int_0^t \frac{1}{2^{2k}}\big\|P_k\nabla_t\xi_i\big\|_{L^2}\cdot\big\|[P_k,\nabla]\xi_i\big\|_{L^2}+\sum_{i=0}^{I}\int_0^t \frac{1}{2^{2k}}\big\|P_k\nabla_t\xi_i\big\|_{L^2}\cdot\big\|[P_k,\psi]\nabla\xi_i\big\|_{L^2}+$$

$$+\sum_{i=0}^{I}\int_0^t \big\|P_k\nabla_t\xi_i\big\|_{L^2}\cdot\big\|[\nabla_t,P_k]\nabla_t\xi_i\big\|_{L^2}+$$

$$+\sum_{i=0}^{I}\int_0^t \frac{1}{2^{2k}}\big\|P_k\nabla_t\xi_i\big\|_{L^2}\cdot\Big(\big\|P_kF_M^i\big\|_{L^2}+\big\|P_k\big(\psi\nabla\big(\nabla^M\Phi_0\big)_Y\big)\big\|_{L^2}\Big)dt'.$$

We note that we dropped the bulk term with a favorable sign in the above estimate. We use Grönwall for $t\in\big[0,1/2\big]$:

$$\sum_{i=0}^{I}\big\|P_k\nabla_t\xi_i\big\|_{L^2}^2+\sum_{i=0}^{I}\frac{1}{2^{2k}}\big\|\nabla P_k\xi_i\big\|_{L^2}^2\lesssim\sum_{i=0}^{I}\frac{1}{2^{2k}}\big\|\nabla P_k\xi_i^0\big\|_{L^2}^2+\sum_{i=0}^{I}\int_0^t\frac{1}{2^{2k}}\big\|\nabla[P_k,\nabla_t]\xi_i\big\|_{L^2}^2+$$

$$+\sum_{i=0}^{I}\int_0^t\frac{1}{2^{2k}}\big\|[\nabla,\nabla_t]P_k\xi_i\big\|_{L^2}^2+\sum_{i=0}^{I}\int_0^t\frac{1}{2^{4k}}\big\|[P_k,\nabla]\xi_i\big\|_{L^2}^2+\sum_{i=0}^{I}\int_0^t\frac{1}{2^{4k}}\big\|[P_k,\psi]\nabla\xi_i\big\|_{L^2}^2+$$

$$+\sum_{i=0}^{I}\int_0^t\big\|[\nabla_t,P_k]\nabla_t\xi_i\big\|_{L^2}^2+\sum_{i=0}^{I}\int_0^t\frac{1}{2^{4k}}\big\|P_kF_i\big\|_{L^2}^2+\int_0^t\frac{1}{2^{4k}}\big\|P_k\big(\psi\nabla\big(\nabla^M\Phi_0\big)_Y\big)\big\|_{L^2}^2.$$

We use Lemma 6.3 in order to bound the commutation terms. Thus, we get:

$$\sum_{i=0}^{I}\big\|P_k\nabla_t\xi_i\big\|_{L^2}^2+\sum_{i=0}^{I}\frac{1}{2^{2k}}\big\|\nabla P_k\xi_i\big\|_{L^2}^2\lesssim\sum_{i=0}^{I}\frac{1}{2^{2k}}\big\|\nabla P_k\xi_i^0\big\|_{L^2}^2+\sum_{i=0}^{I}\int_0^t\frac{1}{2^{6k}}\big\|\xi_i\big\|_{H^1}^2+$$

$$+\sum_{i=0}^{I}\int_0^t\frac{(t')^2}{2^{4k}}\big\|\nabla_t\xi_i\big\|_{L^2}^2+\sum_{i=0}^{I}\int_0^t\frac{1}{2^{4k}}\big\|P_kF_M^i\big\|_{L^2}^2+\int_0^t\frac{1}{2^{4k}}\big\|P_k\big(\psi\nabla\big(\nabla^M\Phi_0\big)_Y\big)\big\|_{L^2}^2.$$

We change variables to τ and get for all $\tau\in[0,2^{-k-1}]$:

$$\sum_{i=0}^{I}\big\|P_k\nabla_\tau\xi_i\big\|_{L^2}^2+\big\|\nabla P_k\xi_i\big\|_{L^2}^2\lesssim\sum_{i=0}^{I}\big\|\nabla P_k\xi_i^0\big\|_{L^2}^2+\sum_{i=0}^{I}\int_0^\tau\frac{1}{2^{3k}}\big\|\xi_i\big\|_{H^1}^2+$$

$$+\sum_{i=0}^{I}\int_0^\tau\frac{(\tau')^2}{2^{k}}\big\|\nabla_\tau\xi_i\big\|_{L^2}^2+\sum_{i=0}^{I}\int_0^\tau\frac{1}{2^{k}}\big\|P_kF_M^i\big\|_{L^2}^2+\int_0^\tau\frac{1}{2^{k}}\big\|P_k\big(\psi\nabla\big(\nabla^M\Phi_0\big)_Y\big)\big\|_{L^2}^2.$$

Next, using Lemma 7.1 and (6.22), we also have the bound for all $\tau\in[0,2^{-k-1}]$:

$$\sum_{i=0}^{I}\big\|P_k\xi_i\big\|_{L^2}^2\lesssim\sum_{i=0}^{I}\big\|P_k\xi_i^0\big\|_{L^2}^2+\sum_{i=0}^{I}2^{-k}\int_0^\tau\big\|\nabla_\tau P_k\xi_i\big\|_{L^2}^2+\sum_{i=0}^{I}2^{k}\int_0^\tau\big\|P_k\xi_i\big\|_{L^2}^2$$

$$\lesssim\sum_{i=0}^{I}\big\|P_k\xi_i^0\big\|_{L^2}^2+\sum_{i=0}^{I}2^{-k}\int_0^\tau\big\|P_k\nabla_\tau\xi_i\big\|_{L^2}^2+$$

$$+\sum_{i=0}^{I} 2^{-5k}\int_0^\tau \|\xi_i\|_{H^1}^2 + \sum_{i=0}^{I} 2^k \int_0^\tau \|P_k\xi_i\|_{L^2}^2.$$

Using Grönwall and the previous estimate, we conclude that:

$$\sum_{i=0}^{I} 2^{2k}\|P_k\xi_i\|_{L^2}^2 \lesssim \sum_{i=0}^{I} 2^{2k}\|P_k\xi_i^0\|_{L^2}^2 + \sum_{i=0}^{I}\|\nabla P_k\xi_i^0\|_{L^2}^2 + \sum_{i=0}^{I}\int_0^\tau \frac{1}{2^{3k}}\|\xi_i\|_{H^1}^2 +$$

$$+\sum_{i=0}^{I}\int_0^\tau \frac{(\tau')^2}{2^k}\|\nabla_\tau\xi_i\|_{L^2}^2 + \sum_{i=0}^{I}\int_0^\tau \frac{1}{2^k}\|P_k F_M^i\|_{L^2}^2 + \int_0^\tau \frac{1}{2^k}\big\|P_k\big(\psi\nabla\big(\nabla^M\Phi_0\big)_Y\big)\big\|_{L^2}^2.$$

□

7.4.2 High Frequency Regime Estimates

We prove a high frequency regime estimate for $P_k\big(\nabla^M\Phi_0\big)_J, P_k\nabla^M\Phi_1,\dots,P_k\nabla^M\Phi_I$, for each $k\geq 0$. The idea of the proof is to use $2^k\tau\nabla_\tau$ as a multiplier for $\tau\in[2^{-k-1},1]$. As in the previous section, our estimates are simplified by the presence of bulk terms with favorable signs. We point out that the argument is entirely analogous to the proof of the high frequency regime estimate for the singular component $P_k\big(\nabla^M\Phi_0\big)_Y$ in Section 7.3.2.

Proposition 7.12. *For any $k\geq 0,\ \tau\in\big[2^{-k-1},1\big]$ we have the estimate for the regular components:*

$$\tau\big\|P_k\nabla_\tau\big(\nabla^M\Phi_0\big)_J\big\|_{L^2}^2 + \frac{1}{\tau}\big\|P_k\big(\nabla^M\Phi_0\big)_J\big\|_{L^2}^2 + \tau\big\|\nabla P_k\big(\nabla^M\Phi_0\big)_J\big\|_{L^2}^2 +$$

$$+\sum_{i=1}^{I}\tau\big\|P_k\nabla_\tau\nabla^M\Phi_i\big\|_{L^2}^2 + \sum_{i=1}^{I}\frac{1}{\tau}\big\|P_k\nabla^M\Phi_i\big\|_{L^2}^2 + \sum_{i=1}^{I}\tau\big\|\nabla P_k\nabla^M\Phi_i\big\|_{L^2}^2 \lesssim$$

$$\lesssim \frac{1}{2^k}\Big(\big\|P_k\nabla_\tau\big(\nabla^M\Phi_0\big)_J\big\|_{L^2}^2 + 2^{2k}\big\|P_k\big(\nabla^M\Phi_0\big)_J\big\|_{L^2}^2 + \big\|\nabla P_k\big(\nabla^M\Phi_0\big)_J\big\|_{L^2}^2\Big)\Big|_{\tau=2^{-k-1}} +$$

$$+\int_{2^{-k-1}}^{\tau}\tau'\big\|\underline{\widetilde{P}}_k\big(\nabla^M\Phi_0\big)_J\big\|_{L^2}^2 + \sum_{i=1}^{I}\int_{2^{-k-1}}^{\tau}\tau'\big\|\underline{\widetilde{P}}_k\nabla^M\Phi_i\big\|_{L^2}^2$$

$$+\sum_{i=1}^{I}\frac{1}{2^k}\Big(\big\|P_k\nabla_\tau\nabla^M\Phi_i\big\|_{L^2}^2 + 2^{2k}\big\|P_k\nabla^M\Phi_i\big\|_{L^2}^2 + \big\|\nabla P_k\nabla^M\Phi_i\big\|_{L^2}^2\Big)\Big|_{\tau=2^{-k-1}}$$

$$+\int_{2^{-k-1}}^{\tau}(\tau')^3\big\|\underline{\widetilde{P}}_k\nabla\big(\nabla^M\Phi_0\big)_J\big\|_{L^2}^2 + \int_{2^{-k-1}}^{\tau}(\tau')^3\big\|\underline{\widetilde{P}}_k\nabla_\tau\big(\nabla^M\Phi_0\big)_J\big\|_{L^2}^2 + \int_{2^{-k-1}}^{\tau}\frac{\tau'}{2^{2k}}\big\|\big(\nabla^M\Phi_0\big)_J\big\|_{H^1}^2$$

$$+\sum_{i=1}^{I}\int_{2^{-k-1}}^{\tau}\Big((\tau')^3\big\|\underline{\widetilde{P}}_k\nabla^{M+1}\Phi_i\big\|_{L^2}^2 + (\tau')^3\big\|\underline{\widetilde{P}}_k\nabla_\tau\nabla^M\Phi_i\big\|_{L^2}^2\Big)d\tau'$$

$$+\sum_{i=1}^{I}\int_{2^{-k-1}}^{\tau}\left(\frac{\tau'}{2^{2k}}\big\|\nabla^M\Phi_i\big\|_{H^1}^2+\frac{(\tau')^3}{2^{2k}}\big\|\nabla_\tau\nabla^M\Phi_i\big\|_{L^2}^2\right)d\tau'$$

$$+\int_{2^{-k-1}}^{\tau}\frac{(\tau')^3}{2^{2k}}\big\|\nabla_\tau\big(\nabla^M\Phi_0\big)_J\big\|_{L^2}^2+\int_{2^{-k-1}}^{\tau}\tau'\big\|P_k\big(\psi\nabla\big(\nabla^M\Phi_0\big)_Y\big)\big\|_{L^2}^2+\sum_{i=0}^{I}\int_{2^{-k-1}}^{\tau}\tau'\big\|P_kF_M^i\big\|_{L^2}^2.$$

Proof. We denote $\xi_0=\big(\nabla^M\Phi_0\big)_J$ and for $1\le i\le I$ we denote $\xi_i=\nabla^M\Phi_i$. We introduce the new time variable $t=2^k\tau$. As before, equations (5.13) and (7.5) with $m=M$ can be written for all $0\le i\le I$:

$$\nabla_t\big(\nabla_t\xi_i\big)+\frac{1}{t}\nabla_t\xi_i-\frac{4}{2^{2k}}\cdot\Delta\xi_i=\sum_{j=0}^{I}\frac{1}{2^{2k}}\cdot\psi\nabla\xi_j+\frac{1}{2^{2k}}\cdot F_M^i+\frac{1}{2^{2k}}\cdot\psi\nabla\big(\nabla^M\Phi_0\big)_Y.$$

We multiply by $\sqrt{t}$ to get for all $0\le i\le I$:

$$\nabla_t\big(\nabla_t(\xi_i\sqrt{t})\big)+\frac{1}{4t^2}\xi_i\sqrt{t}-\frac{4}{2^{2k}}\cdot\Delta\xi_i\sqrt{t}=\sum_{j=0}^{I}\frac{1}{2^{2k}}\cdot\psi\nabla\xi_j\sqrt{t}+\frac{1}{2^{2k}}\cdot F_M^i\sqrt{t}+\frac{1}{2^{2k}}\cdot\psi\nabla\big(\nabla^M\Phi_0\big)_Y\sqrt{t}.$$

For any $k\ge 0$, we apply P_k to obtain the equations for all $0\le i\le I$:

$$\nabla_t\big(P_k\nabla_t(\xi_i\sqrt{t})\big)+\frac{1}{4t^2}P_k\xi_i\sqrt{t}-\frac{4}{2^{2k}}\Delta P_k\xi_i\sqrt{t}=\sum_{j=0}^{I}\frac{1}{2^{2k}}\psi\nabla P_k\xi_j\sqrt{t}+\sum_{j=0}^{I}\frac{1}{2^{2k}}\psi[P_k,\nabla]\xi_j\sqrt{t}+$$

$$+\sum_{j=0}^{I}\frac{1}{2^{2k}}[P_k,\psi]\nabla\xi_j\sqrt{t}+\frac{P_k\big(F_M^i\sqrt{t}+\psi\nabla\big(\nabla^M\Phi_0\big)_Y\sqrt{t}\big)}{2^{2k}}+[\nabla_t,P_k]\nabla_t\xi_i\sqrt{t}$$

We contract each equation with $P_k\nabla_t(\xi_i\sqrt{t})$ and integrate by parts to obtain the following energy estimate:

$$\big\|P_k\nabla_t\xi_i\sqrt{t}\big\|_{L^2}^2+\frac{1}{t^2}\big\|P_k\xi_i\sqrt{t}\big\|_{L^2}^2+\frac{1}{2^{2k}}\big\|\nabla P_k\xi_i\sqrt{t}\big\|_{L^2}^2+\int_{1/2}^{t}\frac{1}{(t')^2}\big\|P_k\xi_i\big\|_{L^2}^2dt'\lesssim$$

$$\lesssim\left(\big\|P_k\nabla_t\xi_i\big\|_{L^2}^2+\big\|P_k\xi_i\big\|_{L^2}^2+\frac{1}{2^{2k}}\big\|\nabla P_k\xi_i\big\|_{L^2}^2\right)\bigg|_{t=1/2}+$$

$$+\int_{1/2}^{t}\int_{S^n}\frac{1}{(t')^2}\big|P_k\xi_i\sqrt{t'}\big|\cdot\big|[P_k,\nabla_t]\xi_i\sqrt{t'}\big|+\int_{1/2}^{t}\int_{S^n}\frac{1}{2^{2k}}\big|\nabla P_k\xi_i\sqrt{t'}\big|\cdot\big|\nabla[P_k,\nabla_t]\xi_i\sqrt{t'}\big|+$$

$$+\int_{1/2}^{t}\int_{S^n}\frac{1}{2^{2k}}\big|\nabla P_k\xi_i\sqrt{t'}\big|\cdot\big|[\nabla,\nabla_t]P_k\xi_i\sqrt{t'}\big|+\sum_{j=0}^{I}\int_{1/2}^{t}\int_{S^n}\frac{1}{2^{2k}}\big|P_k\nabla_t\xi_i\sqrt{t'}\big|\cdot\big|\nabla P_k\xi_j\sqrt{t'}\big|+$$

$$+\sum_{j=0}^{I}\int_{1/2}^{t}\int_{S^n}\frac{1}{2^{2k}}\big|P_k\nabla_t\xi_i\sqrt{t'}\big|\cdot\big|[P_k,\nabla]\xi_j\sqrt{t'}\big|+\sum_{j=0}^{I}\int_{1/2}^{t}\int_{S^n}\frac{1}{2^{2k}}\big|P_k\nabla_t\xi_i\sqrt{t'}\big|\cdot\big|[P_k,\psi]\nabla\xi_j\sqrt{t'}\big|+$$

$$+\int_{1/2}^{t}\int_{S^n}\big|P_k\nabla_t\xi_i\sqrt{t'}\big|\cdot\big|[\nabla_t,P_k]\nabla_t\xi_i\sqrt{t'}\big|+$$

$$+\int_{1/2}^{t}\int_{S^n}\frac{1}{2^{2k}}\big|P_k\nabla_t\xi_i\sqrt{t'}\big|\cdot\big|P_k\big(F_M^i\sqrt{t'}+\psi\nabla\big(\nabla^M\Phi_0\big)_Y\sqrt{t'}\big)\big|.$$

We point out that the bulk term with a favorable sign provides a significant simplification for our analysis. On the other hand, in the analysis of the second model system the corresponding term will create several complications. We use Grönwall for $t\in\big[1/2,2^k\big]$, and the bounds in Lemma 6.4 to get for all $0\le i\le I$:

$$\begin{aligned}
&\big\|P_k\nabla_t\xi_i\sqrt{t}\big\|_{L^2}^2+\frac{1}{t^2}\big\|P_k\xi_i\sqrt{t}\big\|_{L^2}^2+\frac{1}{2^{2k}}\big\|\nabla P_k\xi_i\sqrt{t}\big\|_{L^2}^2\lesssim\\
&\lesssim\bigg(\big\|P_k\nabla_t\xi_i\big\|_{L^2}^2+\big\|P_k\xi_i\big\|_{L^2}^2+\frac{1}{2^{2k}}\big\|\nabla P_k\xi_i\big\|_{L^2}^2\bigg)\bigg|_{t=1/2}\\
&+\int_{1/2}^{t}\frac{1}{2^{2k}t'}\big\|P_k\xi_i\sqrt{t'}\big\|_{L^2}\cdot\Big(\big\|\widetilde{\underline{P}}_k\xi_i\sqrt{t'}\big\|_{L^2}+2^{-k}\big\|\xi_i\sqrt{t'}\big\|_{L^2}\Big)+\\
&+\sum_{j=0}^{I}\int_{1/2}^{t}\frac{1}{2^{2k}}\big\|P_k\nabla_t\xi_i\sqrt{t'}\big\|_{L^2}\cdot\big\|\nabla P_k\xi_j\sqrt{t'}\big\|_{L^2}+\\
&+\int_{1/2}^{t}\frac{t'}{2^{4k}}\big\|\nabla P_k\xi_i\sqrt{t'}\big\|_{L^2}\cdot\Big(\big\|\widetilde{\underline{P}}_k\nabla\xi_i\sqrt{t'}\big\|_{L^2}+2^{-k}\big\|\xi_i\sqrt{t'}\big\|_{H^1}\Big)+\\
&+\sum_{j=0}^{I}\int_{1/2}^{t}\frac{1}{2^{3k}}\big\|P_k\nabla_t\xi_i\sqrt{t'}\big\|_{L^2}\cdot\big\|\xi_j\sqrt{t'}\big\|_{H^1}+\\
&+\int_{1/2}^{t}\frac{t'}{2^{2k}}\big\|P_k\nabla_t\xi_i\sqrt{t'}\big\|_{L^2}\Big(\big\|\widetilde{\underline{P}}_k\nabla_t\xi_i\sqrt{t'}\big\|_{L^2}+2^{-k}\big\|\nabla_t\xi_i\sqrt{t'}\big\|_{L^2}\Big)+\\
&+\int_{1/2}^{t}\frac{1}{2^{2k}}\big\|P_k\nabla_t\xi_i\sqrt{t'}\big\|_{L^2}\big\|P_kF_M^i\sqrt{t'}\big\|_{L^2}+\int_{1/2}^{t}\frac{1}{2^{2k}}\big\|P_k\nabla_t\xi_i\sqrt{t'}\big\|_{L^2}\big\|P_k\big(\psi\nabla\big(\nabla^M\Phi_0\big)_Y\big)\sqrt{t'}\big\|_{L^2}
\end{aligned}$$

Once again we use Grönwall for $t\in\big[1/2,2^k\big]$ to get for all $0\le i\le I$:

$$\begin{aligned}
&\big\|P_k\nabla_t\xi_i\sqrt{t}\big\|_{L^2}^2+\frac{1}{t^2}\big\|P_k\xi_i\sqrt{t}\big\|_{L^2}^2+\frac{1}{2^{2k}}\big\|\nabla P_k\xi_i\sqrt{t}\big\|_{L^2}^2\lesssim\\
&\lesssim\bigg(\big\|P_k\nabla_t\xi_i\big\|_{L^2}^2+\big\|P_k\xi_i\big\|_{L^2}^2+\frac{1}{2^{2k}}\big\|\nabla P_k\xi_i\big\|_{L^2}^2\bigg)\bigg|_{t=1/2}+\\
&+\int_{1/2}^{t}\frac{1}{2^{3k}}\big\|\widetilde{\underline{P}}_k\xi_i\sqrt{t'}\big\|_{L^2}^2+\int_{1/2}^{t}\frac{(t')^2}{2^{5k}}\big\|\widetilde{\underline{P}}_k\nabla\xi_i\sqrt{t'}\big\|_{L^2}^2+\int_{1/2}^{t}\frac{(t')^2}{2^{3k}}\big\|\widetilde{\underline{P}}_k\nabla_t\xi_i\sqrt{t'}\big\|_{L^2}^2+\\
&+\sum_{j=0}^{I}\int_{1/2}^{t}\frac{1}{2^{3k}}\big\|\nabla P_k\xi_j\sqrt{t'}\big\|_{L^2}^2+\sum_{j=0}^{I}\int_{1/2}^{t}\frac{1}{2^{5k}}\big\|\xi_j\sqrt{t'}\big\|_{H^1}^2+\int_{1/2}^{t}\frac{(t')^2}{2^{5k}}\big\|\nabla_t\xi_i\sqrt{t'}\big\|_{L^2}^2+\\
&+\int_{1/2}^{t}\frac{1}{2^{3k}}\big\|P_kF_M^i\sqrt{t'}\big\|_{L^2}^2+\int_{1/2}^{t}\frac{1}{2^{3k}}\big\|P_k\big(\psi\nabla\big(\nabla^M\Phi_0\big)_Y\big)\sqrt{t'}\big\|_{L^2}^2.
\end{aligned}$$

Finally, we sum the above estimates for all $0 \leq i \leq I$ to obtain:

$$\sum_{i=0}^{I} t \big\| P_k \nabla_t \xi_i \big\|_{L^2}^2 + \sum_{i=0}^{I} \frac{1}{t} \big\| P_k \xi_i \big\|_{L^2}^2 + \sum_{i=0}^{I} \frac{t}{2^{2k}} \big\| \nabla P_k \xi_i \big\|_{L^2}^2 \lesssim$$

$$\lesssim \sum_{i=0}^{I} \Big(\big\| P_k \nabla_t \xi_i \big\|_{L^2}^2 + \big\| P_k \xi_i \big\|_{L^2}^2 + \frac{1}{2^{2k}} \big\| \nabla P_k \xi_i \big\|_{L^2}^2 \Big) \Big|_{t=1/2} + \sum_{i=0}^{I} \int_{1/2}^{t} \frac{1}{2^{3k}} \big\| \widetilde{\underline{P}}_k \xi_i \sqrt{t'} \big\|_{L^2}^2 +$$

$$+ \sum_{i=0}^{I} \int_{1/2}^{t} \frac{(t')^2}{2^{5k}} \big\| \widetilde{\underline{P}}_k \nabla \xi_i \sqrt{t'} \big\|_{L^2}^2 + \sum_{i=0}^{I} \int_{1/2}^{t} \frac{(t')^2}{2^{3k}} \big\| \widetilde{\underline{P}}_k \nabla_t \xi_i \sqrt{t'} \big\|_{L^2}^2 + \sum_{i=0}^{I} \int_{1/2}^{t} \frac{1}{2^{5k}} \big\| \xi_i \sqrt{t'} \big\|_{H^1}^2 +$$

$$+ \sum_{i=0}^{I} \int_{1/2}^{t} \frac{(t')^2}{2^{5k}} \big\| \nabla_t \xi_i \sqrt{t'} \big\|_{L^2}^2 + \sum_{i=0}^{I} \int_{1/2}^{t} \frac{t'}{2^{3k}} \big\| P_k F_M^i \big\|_{L^2}^2 + \int_{1/2}^{t} \frac{t'}{2^{3k}} \big\| P_k \big(\psi \nabla \big(\nabla^M \Phi_0 \big)_Y \big) \big\|_{L^2}^2 .$$

We change variables to τ in order to obtain the conclusion. $\square$

7.4.3 The Proof of Theorem 7.1

In this section, we combine the low frequency regime and the high frequency regime estimates for the regular components to establish top order estimates. Together with (7.7), (7.8), and the lower order estimates, these complete the proof of Theorem 7.1.

We recall the definitions of $\mathcal{E}_I$, $\mathcal{D}_I$, and $\mathcal{F}_I$ in the statement of Theorem 7.1:

$$\begin{aligned} \mathcal{E}_I(\tau) = & \tau^2 \big\| \nabla_\tau \nabla^M \Phi_0 \big\|_{H^{1/2}(S_\tau)}^2 + \tau^2 \big\| \nabla^M \Phi_0 \big\|_{H^{3/2}(S_\tau)}^2 + \\ & + \sum_{i=1}^{I} \Big(\tau \big\| \nabla_\tau \nabla^M \Phi_i \big\|_{H^{1/2}(S_\tau)}^2 + \tau \big\| \nabla^{M+1} \Phi_i \big\|_{H^{1/2}(S_\tau)}^2 + \big\| \Phi_i \big\|_{H^{M+1}(S_\tau)}^2 \Big), \end{aligned}$$

$$\mathcal{D}_I = \big\| \mathcal{O} \big\|_{H^{M+1}(S^n)}^2 + \big\| \mathfrak{h} \big\|_{H^{M+1}(S^n)}^2 + \sum_{i=1}^{I} \big\| \Phi_i^0 \big\|_{H^{M+1}(S^n)}^2 ,$$

$$\mathcal{F}_I(\tau) = \sum_{m=0}^{M} \sum_{i=0}^{I} \int_0^\tau \big\| F_m^i \big\|_{L^2(S_{\tau'})}^2 d\tau' + \sum_{i=0}^{I} \int_0^\tau \tau' \big\| F_M^i \big\|_{H^{1/2}(S_{\tau'})}^2 d\tau' .$$

Proof of Theorem 7.1. Using (7.7), (7.8), and the lower order estimates in Propositions 7.2 and 7.3, we notice that in order to establish Theorem 7.1 it suffices to prove the estimate for all $\tau \in (0, 1]$:

$$\tau \big\| \nabla_\tau \big(\nabla^M \Phi_0 \big)_J \big\|_{H^{1/2}}^2 + \tau \big\| \nabla \big(\nabla^M \Phi_0 \big)_J \big\|_{H^{1/2}}^2 + \sum_{i=1}^{I} \tau \big\| \nabla_\tau \nabla^M \Phi_i \big\|_{H^{1/2}}^2 +$$

$$+ \sum_{i=1}^{I} \tau \big\| \nabla^{M+1} \Phi_i \big\|_{H^{1/2}}^2 + \big\| \big(\nabla^M \Phi_0 \big)_J \big\|_{H^1}^2 + \sum_{i=1}^{I} \big\| \nabla^M \Phi_i \big\|_{H^1}^2$$

$$\lesssim \|\mathcal{O}\|_{H^{M+1}}^2 + \|\mathfrak{h}\|_{H^{M+1}}^2 + \sum_{i=1}^{I} \|\Phi_i^0\|_{H^{M+1}}^2 + \sum_{i=0}^{I} \int_0^\tau \|F_M^i\|_{L^2}^2 + \tau' \|F_M^i\|_{H^{1/2}}^2 d\tau'.$$

For the rest of the proof we show this estimate. The first step is to combine the high frequency regime estimate in Proposition 7.12 with the low frequency regime estimate in Proposition 7.11. We denote $\xi_0 = \big(\nabla^M \Phi_0\big)_J$ and $\xi_i = \nabla^M \Phi_i$ for $1 \leq i \leq I$. Using the high frequency regime estimate in Proposition 7.12, we get that for $k \geq 0$ and $\tau \in \big[2^{-k-1}, 1\big]$:

$$\sum_{i=0}^{I} 2^k \tau \big\|P_k \nabla_\tau \xi_i\big\|_{L^2}^2 + \sum_{i=0}^{I} \frac{2^k}{\tau} \big\|P_k \xi_i\big\|_{L^2}^2 + \sum_{i=0}^{I} 2^k \tau \big\|\nabla P_k \xi_i\big\|_{L^2}^2 \lesssim$$

$$\lesssim \sum_{i=0}^{I} \bigg(\big\|P_k \nabla_\tau \xi_i\big\|_{L^2}^2 + 2^{2k} \big\|P_k \xi_i\big\|_{L^2}^2 + \big\|\nabla P_k \xi_i\big\|_{L^2}^2 \bigg)\bigg|_{\tau = 2^{-k-1}} + \sum_{i=0}^{I} \int_{2^{-k-1}}^{\tau} 2^k \tau' \big\|\underline{\widetilde{P}}_k \xi_i\big\|_{L^2}^2$$

$$+ \sum_{i=0}^{I} \int_{2^{-k-1}}^{\tau} 2^k (\tau')^3 \big\|\underline{\widetilde{P}}_k \nabla \xi_i\big\|_{L^2}^2 + \sum_{i=0}^{I} \int_{2^{-k-1}}^{\tau} 2^k (\tau')^3 \big\|\underline{\widetilde{P}}_k \nabla_\tau \xi_i\big\|_{L^2}^2 + \sum_{i=0}^{I} \int_{2^{-k-1}}^{\tau} \frac{\tau'}{2^k} \big\|\xi_i\big\|_{H^1}^2$$

$$+ \sum_{i=0}^{I} \int_{2^{-k-1}}^{\tau} \frac{(\tau')^3}{2^k} \big\|\nabla_\tau \xi_i\big\|_{L^2}^2 + \sum_{i=0}^{I} \int_{2^{-k-1}}^{\tau} 2^k \tau' \big\|P_k F_M^i\big\|_{L^2}^2 + \int_{2^{-k-1}}^{\tau} 2^k \tau' \Big\|P_k \big(\psi \nabla \big(\nabla^M \Phi_0\big)_Y\big)\Big\|_{L^2}^2.$$

We use the low frequency regime estimate in Proposition 7.11 and the bound (7.8) for the singular component. Thus, we get for $k \geq 0$ and $\tau \in \big[2^{-k-1}, 1\big]$ the following high frequency regime estimate:

$$\sum_{i=0}^{I} 2^k \tau \big\|P_k \nabla_\tau \xi_i\big\|_{L^2}^2 + \sum_{i=0}^{I} \frac{2^k}{\tau} \big\|P_k \xi_i\big\|_{L^2}^2 + \sum_{i=0}^{I} 2^k \tau \big\|\nabla P_k \xi_i\big\|_{L^2}^2 \lesssim$$

$$\lesssim \sum_{i=0}^{I} \big\|\nabla P_k \xi_i^0\big\|_{L^2}^2 + \sum_{i=0}^{I} 2^{2k} \big\|P_k \xi_i^0\big\|_{L^2}^2 + 2^{-k} \big\|\mathcal{O}\big\|_{H^{M+1}}^2 + \sum_{i=0}^{I} \int_0^{2^{-k-1}} \frac{1}{2^{3k}} \big\|\xi_i\big\|_{H^1}^2 +$$

$$+ \sum_{i=0}^{I} \int_0^{2^{-k-1}} \frac{(\tau')^2}{2^k} \big\|\nabla_\tau \xi_i\big\|_{L^2}^2 + \sum_{i=0}^{I} \int_0^{2^{-k-1}} \frac{1}{2^k} \big\|P_k F_M^i\big\|_{L^2}^2 + \sum_{i=0}^{I} \int_{2^{-k-1}}^{\tau} \frac{\tau'}{2^k} \big\|\xi_i\big\|_{H^1}^2$$

$$+ \sum_{i=0}^{I} \int_{2^{-k-1}}^{\tau} \bigg(2^k \tau' \big\|\underline{\widetilde{P}}_k \xi_i\big\|_{L^2}^2 + 2^k (\tau')^3 \big\|\underline{\widetilde{P}}_k \nabla \xi_i\big\|_{L^2}^2 + 2^k (\tau')^3 \big\|\underline{\widetilde{P}}_k \nabla_\tau \xi_i\big\|_{L^2}^2 \bigg) d\tau'$$

$$+ \sum_{i=0}^{I} \int_{2^{-k-1}}^{\tau} \frac{(\tau')^3}{2^k} \big\|\nabla_\tau \xi_i\big\|_{L^2}^2 + \sum_{i=0}^{I} \int_{2^{-k-1}}^{\tau} 2^k \tau' \big\|P_k F_M^i\big\|_{L^2}^2 + \int_{2^{-k-1}}^{\tau} 2^k \tau' \big\|\nabla P_k \big(\nabla^M \Phi_0\big)_Y\big\|_{L^2}^2.$$

The second step is to prove a bound for the sum of the non-negative frequencies. We define the following energy for all $k \geq 0$:

$$2^k E_k^2(\tau) = \sum_{i=0}^{I} 2^k \tau \big\|P_k \nabla_\tau \xi_i\big\|_{L^2}^2 + \sum_{i=0}^{I} 2^k \big\|P_k \xi_i\big\|_{L^2}^2 + \sum_{i=0}^{I} 2^k \tau \big\|\nabla P_k \xi_i\big\|_{L^2}^2 + \sum_{i=0}^{I} \big\|\nabla P_k \xi_i\big\|_{L^2}^2.$$

Using the singular component high frequency regime estimate, we get that for all $k \geq 0$:

$$\sum_{\tau>2^{-k-1}} \int_{2^{-k-1}}^{\tau} 2^k\tau' \big\|\nabla P_k\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2 \lesssim \int_0^\tau \sum_{\tau'>2^{-k-1}} 2^k\tau'\big\|\nabla P_k\big(\nabla^M\Phi_0\big)_Y\big\|_{L^2}^2 d\tau' \lesssim \big\|\mathcal{O}\big\|_{H^{M+1}}^2.$$

As a result, the above high frequency regime estimate implies the bound:

$$\sum_{\tau>2^{-k-1}} 2^k E_k^2 \lesssim \big\|\mathcal{O}\big\|_{H^{M+1}}^2 + \sum_{i=0}^{I}\big\|\xi_i^0\big\|_{H^1}^2 + \sum_{i=0}^{I}\int_0^\tau \big\|\xi_i\big\|_{H^1}^2 + \sum_{i=0}^{I}\int_0^\tau (\tau')^2\big\|\nabla_\tau\xi_i\big\|_{H^{1/2}}^2 +$$

$$+\sum_{i=0}^{I}\int_0^\tau (\tau')^2\big\|\nabla\xi_i\big\|_{H^{1/2}}^2 + \int_0^\tau \tau'\big\|\xi_i\big\|_{H^{1/2}}^2 + \sum_{i=0}^{I}\sum_{k\geq 0}\int_0^\tau \big(2^k\tau'+2^{-k}\big)\big\|P_kF_M^i\big\|_{L^2}^2.$$

The low frequency estimates in Proposition 7.3 and (7.8) imply the bound:

$$\sum_{\tau\leq 2^{-k-1}} 2^k E_k^2 \lesssim \big\|\mathcal{O}\big\|_{H^{M+1}}^2 + \sum_{i=0}^{I}\big\|\xi_i^0\big\|_{H^1}^2 +$$

$$+\sum_{i=0}^{I}\int_0^\tau \big\|\xi_i\big\|_{H^1}^2 + \sum_{i=0}^{I}\int_0^\tau (\tau')^2\big\|\nabla_\tau\xi_i\big\|_{L^2}^2 + \sum_{i=0}^{I}\sum_{k\geq 0}\int_0^\tau \frac{1}{2^k}\big\|P_kF_M^i\big\|_{L^2}^2.$$

As a result, we completed the second step of the proof and showed that:

$$\sum_{k\geq 0} 2^k E_k^2 \lesssim \big\|\mathcal{O}\big\|_{H^{M+1}}^2 + \sum_{i=0}^{I}\big\|\xi_i^0\big\|_{H^1}^2 + \sum_{i=0}^{I}\int_0^\tau \big\|\xi_i\big\|_{H^1}^2 + \sum_{i=0}^{I}\int_0^\tau (\tau')^2\big\|\nabla_\tau\xi_i\big\|_{H^{1/2}}^2 +$$

$$+\sum_{i=0}^{I}\int_0^\tau (\tau')^2\big\|\nabla\xi_i\big\|_{H^{1/2}}^2 + \sum_{i=0}^{I}\int_0^\tau \big\|F_M^i\big\|_{L^2}^2 + \tau'\big\|F_M^i\big\|_{H^{1/2}}^2 d\tau'.$$

The final step of the proof is dealing with the negative frequencies $k<0$. We remark that we can repeat the proof of Proposition 7.3 for $m=M$ and use (7.8) in order to deal with the singular component. Thus, to get:

$$\sum_{i=0}^{I}\big\|\nabla_\tau\xi_i\big\|_{L^2}^2 + \sum_{i=0}^{I}\big\|\xi_i\big\|_{H^1}^2 \lesssim \big\|\mathcal{O}\big\|_{H^{M+1}}^2 + \sum_{i=0}^{I}\big\|\xi_i^0\big\|_{H^1}^2 + \sum_{i=0}^{I}\int_0^\tau \big\|F_M^i\big\|_{L^2}^2 d\tau'.$$

We notice that by [KR06], we have that $\|P_k\nabla F\|_{L^2} \lesssim 2^k\|F\|_{L^2}$ for any $k<0$. We obtain the following bound for the negative frequencies:

$$\sum_{i=0}^{I}\sum_{k<0}\Big(2^k\tau\big\|P_k\nabla_\tau\xi_i\big\|_{L^2}^2 + 2^k\big\|P_k\xi_i\big\|_{L^2}^2 + 2^k\tau\big\|\nabla P_k\xi_i\big\|_{L^2}^2 + \big\|\nabla P_k\xi_i\big\|_{L^2}^2\Big) \lesssim \sum_{i=0}^{I}\big\|\xi_i\big\|_{L^2}^2 + \big\|\nabla_\tau\xi_i\big\|_{L^2}^2$$

$$\lesssim \big\|\mathcal{O}\big\|_{H^{M+1}}^2 + \sum_{i=0}^{I}\big\|\xi_i^0\big\|_{H^1}^2 + \sum_{i=0}^{I}\int_0^\tau \big\|F_M^i\big\|_{L^2}^2 d\tau'.$$

To conclude the proof of Theorem 7.1, we combine the estimates proved for non-negative frequencies and negative frequencies. We then apply Grönwall to obtain:

$$\sum_{i=0}^{I} \tau \big\|\nabla_\tau \xi_i\big\|^2_{H^{1/2}} + \sum_{i=0}^{I} \big\|\xi_i\big\|^2_{H^1} + \sum_{i=0}^{I} \tau \big\|\nabla \xi_i\big\|^2_{H^{1/2}} \lesssim$$

$$\lesssim \big\|\mathcal{O}\big\|^2_{H^{M+1}} + \sum_{i=0}^{I} \big\|\xi_i^0\big\|^2_{H^1} + \sum_{i=0}^{I} \int_0^\tau \big\|F^i_M\big\|^2_{L^2} + \tau' \big\|F^i_M\big\|^2_{H^{1/2}}.$$

□

8 Estimates for the Second Model System

An essential part of our argument is to prove estimates on the second model system (5.14), in terms of the initial data at $\{\tau = 1\}$. This system includes the commuted Bianchi system, where Φ_0 and Φ_i correspond to the commuted curvature components $\nabla_4^{\frac{n-4}{2}} \alpha$ and $\nabla_4^{\frac{n-4}{2}} \Psi^G$.

In the present section, we prove in Theorem 8.1 the main estimates for the system (5.14). This section is based on [Cic24, Section 9] and [Cic26, Section 4]. We encourage the reader to return to Section 1.3.3 of the introduction for an outline of the proof.

Theorem 8.1. *Let $M > N$ be large enough. We assume that $\Phi_0, \ldots, \Phi_I$ satisfy the second model system (5.14) on the hypersurface $\{u = -1\} \times \{\tau \in (0,1)\} \times S^n$ of the spacetime $\big(\mathcal{M}, g\big)$ obtained in Theorem 3.1.*

For all $\tau \in (0,1]$, we denote by $\mathcal{E}_{II}(\tau)$ the energy of the solution at time τ:

$$\begin{aligned}\mathcal{E}_{II}(\tau) =& \tau \big\|\Phi_0\big\|^2_{H^{M+1/2}(S_\tau)} + \tau^2 \big\|\Phi_0\big\|^2_{H^{M+3/2}(S_\tau)} + \tau^2 \big\|\nabla_\tau \nabla^M \Phi_0\big\|^2_{H^{1/2}(S_\tau)} + \\ &+ \sum_{m=0}^{M-1} \tau^2 \big\|\nabla_\tau \nabla^m \Phi_0\big\|^2_{L^2(S_\tau)} + \int_\tau^1 \tau' \big\|\Phi_0\big\|^2_{H^{M+1}} d\tau' + \sum_{i=1}^{I} \big\|\Phi_i\big\|^2_{H^{M+3/2}(S_\tau)} + \\ &+ \sum_{i=1}^{I} \sum_{m=0}^{M} \Big(\big\|\nabla_\tau \nabla^m \Phi_i\big\|^2_{H^{1/2}(S_\tau)} + \int_\tau^1 \frac{1}{\tau'} \big\|\nabla_\tau \nabla^m \Phi_i\big\|^2_{H^{1/2}} d\tau' \Big).\end{aligned} \tag{8.1}$$

We define the initial data norm and the inhomogeneous norm as:

$$\mathcal{D}_{II} = \sum_{i=0}^{I} \big\|\Phi_i\big\|^2_{H^{M+3/2}(S_1)} + \sum_{i=0}^{I} \big\|\nabla_\tau \Phi_i\big\|^2_{H^{M+1/2}(S_1)}, \ \mathcal{F}_{II}(\tau) = \sum_{i=0}^{I} \sum_{m=0}^{M} \int_\tau^1 \tau' \big\|F^i_m\big\|^2_{H^{1/2}(S_{\tau'})} d\tau'.$$

The solution of the second model system satisfies the estimate for $\tau \in (0,1]$ and a constant $C_{II} > 0$:

$$\mathcal{E}_{II}(\tau) \leq C_{II} \mathcal{D}_{II} + C_{II} \mathcal{F}_{II}(\tau), \tag{8.2}$$

where the constant C_{II} depends on $M > N$, the bound on $\|\psi\|_{H^{M+1}}\big|_{\tau=1}$, and the bounds satisfied by the background $\big(\mathcal{M}, g\big)$ according to Theorem 3.1.

Moreover, the asymptotic data at $\mathcal{I}^-$ given by Φ_i^0, $\mathcal{O}$, and $\mathfrak{h}$ satisfies the estimates:

$$\big\|\mathcal{O}\big\|^2_{H^{M+1}(S^n)} + \sum_{i=1}^{I} \big\|\Phi_i^0\big\|^2_{H^{M+1}(S^n)} \leq C_{II}\mathcal{D}_{II} + C_{II}\mathcal{F}_{II}(0), \tag{8.3}$$

$$\big\|\mathfrak{h}\big\|^2_{H^{M+1}(S^n)} \leq C_{II} \cdot \Big(\mathcal{D}_{II} + \mathcal{F}_{II}(0) + \big\|h\big\|^2_{L^2(S^n)}\Big). \tag{8.4}$$

We outline the structure of the rest of the section. In Section 8.1, we prove estimates for the regular quantities $\Phi_1, \ldots, \Phi_I$ which satisfy (5.14) and decouple from the singular quantity Φ_0. We prove (1.44) and (1.45) in Proposition 8.1. In Section 8.2, we prove the preliminary bound (1.46) in Proposition 8.2. In Section 8.3, we prove the low frequency regime estimates in Propositions 8.3 and 8.4. In Section 8.4, we prove the preliminary high frequency regime estimate in Proposition 8.5. We improve this in Section 8.5 to obtain the high frequency regime estimate in Proposition 8.6. Combining the previous estimates in Section 8.6, we complete the proof of the main estimate (8.2) in Theorem 8.1. Finally, we establish the estimates (8.3) and (8.4) for the asymptotic data at $\mathcal{I}^-$ in Section 8.7. We prove (1.48) in Proposition 8.7 and (1.49) in Proposition 8.8.

We point out that the main difficulty in our argument is dealing with the top order quantity $\xi = \nabla^M \Phi_0$. For this part we use as a guideline the toy problem considered in [Cic24, Section 9], where we studied the equation satisfied by $\xi = \nabla^M \Phi_0$, but we dropped the terms $F'_M = F^0_M + \sum_{i=1}^{I} \psi \nabla^{M+1} \Phi_i$ for simplicity.

8.1 Estimates for the Regular Quantities

In this section, we prove the main estimates for all the regular quantities Φ_i, with $1 \leq i \leq I$:

Proposition 8.1. *For all $0 \leq m \leq M$ and $1 \leq i \leq I$ we have:*

$$\big\|\nabla_\tau \nabla^m \Phi_i\big\|^2_{H^{1/2}} + \big\|\nabla^m \Phi_i\big\|^2_{H^{3/2}} + \int_\tau^1 \frac{1}{\tau'} \big\|\nabla_\tau \nabla^m \Phi_i\big\|^2_{H^{1/2}} d\tau' \lesssim$$

$$\lesssim \sum_{j=1}^{I} \Bigg(\big\|\Phi_j\big\|^2_{H^{m+3/2}}\big|_{\tau=1} + \big\|\nabla_\tau \Phi_j\big\|^2_{H^{m+1/2}}\big|_{\tau=1} + \sum_{k=0}^{m} \int_\tau^1 \tau' \big\|F_k^j\big\|^2_{H^{1/2}} d\tau' \Bigg) \lesssim \mathcal{D}_{II} + \mathcal{F}_{II}(\tau).$$

Remark 8.1. *As pointed out in Section 1.3.3 of the Introduction, we recall that the second model system* (5.14) *has a favorable structure for proving estimates backwards in time for the regular* *quantities $\Phi_1, \dots, \Phi_I$. In particular, when using the ∇_τ multiplier we obtain good bulk terms, allowing us to prove top order estimates.*

Proof. For the purpose of this proof we fix $0 \leq m \leq M$ and we denote $\xi_i = \nabla^m \Phi_i$. We can rewrite (5.14) for all $1 \leq i \leq I$:

$$\nabla_\tau\big(\nabla_\tau \xi_i\big) - \frac{1}{\tau}\nabla_\tau \xi_i - 4\Delta \xi_i = \sum_{j=1}^{I} \psi \nabla \xi_j + F_m^i. \tag{8.5}$$

Preliminary estimates. We first prove the following preliminary estimates for all $1 \leq i \leq I$:

$$\big\|\nabla_\tau \nabla^m \Phi_i\big\|_{L^2}^2 + \big\|\nabla^m \Phi_i\big\|_{H^1}^2 + \int_\tau^1 \frac{1}{\tau'}\big\|\nabla_\tau \nabla^m \Phi_i\big\|_{L^2}^2 d\tau' \lesssim$$
$$\lesssim \sum_{j=1}^{I}\Big(\big\|\Phi_j\big\|_{H^{m+1}}^2\big|_{\tau=1} + \big\|\nabla_\tau \Phi_j\big\|_{H^m}^2\big|_{\tau=1} + \int_\tau^1 \tau'\big\|F_m^j\big\|_{L^2}^2\Big).$$

To prove this, we contract each equation (8.5) by $\nabla_\tau \xi_i$. Using Lemma 7.1, we obtain the energy estimate:

$$\big\|\nabla_\tau \xi_i\big\|_{L^2}^2 + \big\|\nabla \xi_i\big\|_{L^2}^2 + \int_\tau^1 \frac{1}{\tau'}\big\|\nabla_\tau \xi_i\big\|_{L^2}^2 d\tau' \lesssim \Big(\big\|\xi_i\big\|_{H^1}^2 + \big\|\nabla_\tau \xi_i\big\|_{L^2}^2\Big)\Big|_{\tau=1} +$$
$$+ \int_\tau^1 \tau'\big\|\nabla \xi_i\big\|_{L^2}\big\|[\nabla, \nabla_4]\xi_i\big\|_{L^2} d\tau' + \sum_{j=1}^{I}\int_\tau^1 \big\|\nabla \xi_j\big\|_{L^2}\big\|\nabla_\tau \xi_i\big\|_{L^2} + \int_\tau^1 \big\|\nabla_\tau \xi_i\big\|_{L^2}\big\|F_m^i\big\|_{L^2} d\tau'.$$

We use Cauchy-Schwarz, Grönwall, and the bulk term to obtain:

$$\big\|\nabla_\tau \xi_i\big\|_{L^2}^2 + \big\|\nabla \xi_i\big\|_{L^2}^2 + \int_\tau^1 \frac{1}{\tau'}\big\|\nabla_\tau \xi_i\big\|_{L^2}^2 d\tau' \lesssim \Big(\big\|\xi_i\big\|_{H^1}^2 + \big\|\nabla_\tau \xi_i\big\|_{L^2}^2\Big)\Big|_{\tau=1} +$$
$$+ \int_\tau^1 (\tau')^2\big\|\xi_i\big\|_{L^2}^2 d\tau' + \sum_{j=1}^{I}\int_\tau^1 \big\|\nabla \xi_j\big\|_{L^2}^2 d\tau' + \int_\tau^1 \tau'\big\|F_m^i\big\|_{L^2}^2 d\tau'.$$

On the other hand, we also have the bound:

$$\big\|\xi_i\big\|_{L^2}^2 \lesssim \big\|\xi_i\big\|_{L^2}^2\Big|_{\tau=1} + \int_\tau^1 \big\|\xi_i\big\|_{L^2}\big\|\nabla_\tau \xi_i\big\|_{L^2} d\tau' \lesssim \big\|\xi_i\big\|_{L^2}^2\Big|_{\tau=1} + \int_\tau^1 \big\|\xi_i\big\|_{L^2}^2 d\tau' + \int_\tau^1 \big\|\nabla_\tau \xi_i\big\|_{L^2}^2 d\tau'.$$

We sum the last two inequalities for all $1 \leq i \leq I$:

$$\sum_{i=1}^{I}\big\|\nabla_\tau \xi_i\big\|_{L^2}^2 + \sum_{i=1}^{I}\big\|\xi_i\big\|_{H^1}^2 + \sum_{i=1}^{I}\int_\tau^1 \frac{1}{\tau'}\big\|\nabla_\tau \xi_i\big\|_{L^2}^2 d\tau' \lesssim \sum_{i=1}^{I}\Big(\big\|\xi_i\big\|_{H^1}^2 + \big\|\nabla_\tau \xi_i\big\|_{L^2}^2\Big)\Big|_{\tau=1} +$$
$$+ \sum_{i=1}^{I}\int_\tau^1 \big\|\xi_i\big\|_{H^1}^2 d\tau' + \sum_{i=1}^{I}\int_\tau^1 \big\|\nabla_\tau \xi_i\big\|_{L^2}^2 d\tau' + \sum_{i=1}^{I}\int_\tau^1 \tau'\big\|F_m^i\big\|_{L^2}^2 d\tau'.$$

We use Grönwall to complete the proof of the above preliminary estimate. For the data terms at $\tau = 1$ we also use the commutation formula (6.12) and the assumption on the background spacetime $\|\psi\|_{H^{M+1}(S_1)} \leq C_2$.

The main estimates. For each $k \geq 0$, we apply P_k to equation (8.5) to obtain:

$$\nabla_\tau\big(P_k\nabla_\tau\xi_i\big) - \frac{1}{\tau}P_k\nabla_\tau\xi_i - 4\Delta P_k\xi_i = \sum_{j=1}^{I} P_k\big(\psi\nabla\xi_j\big) + P_kF^i_m + [\nabla_\tau, P_k]\nabla_\tau\xi_i.$$

We contract each equation by $P_k\nabla_\tau\xi_i$, in order to obtain the energy estimate:

$$\big\|P_k\nabla_\tau\xi_i\big\|^2_{L^2} + \big\|\nabla P_k\xi_i\big\|^2_{L^2} + \int_\tau^1 \frac{1}{\tau'}\big\|P_k\nabla_\tau\xi_i\big\|^2_{L^2}d\tau' \lesssim \Big(\big\|\nabla P_k\xi_i\big\|^2_{L^2} + \big\|P_k\nabla_\tau\xi_i\big\|^2_{L^2}\Big)\Big|_{\tau=1}+$$

$$+\int_\tau^1 \tau'\big\|\nabla P_k\xi_i\big\|_{L^2}\big\|\nabla[P_k,\nabla_4]\xi_i\big\|_{L^2}d\tau' + \int_\tau^1 \tau'\big\|\nabla P_k\xi_i\big\|_{L^2}\big\|[\nabla,\nabla_4]P_k\xi_i\big\|_{L^2}d\tau'$$

$$+\sum_{j=1}^{I}\int_\tau^1 \big\|P_k(\psi\nabla\xi_j)\big\|_{L^2}\big\|P_k\nabla_\tau\xi_i\big\|_{L^2} + \int_\tau^1 \big\|P_k\nabla_\tau\xi_i\big\|_{L^2}\big\|P_kF^i_m\big\|_{L^2}d\tau'+$$

$$+\int_\tau^1 \big\|P_k\nabla_\tau\xi_i\big\|_{L^2}\big\|[\nabla_\tau,P_k]\nabla_\tau\xi_i\big\|_{L^2}d\tau'.$$

We use Cauchy-Schwarz and Grönwall to obtain:

$$\big\|P_k\nabla_\tau\xi_i\big\|^2_{L^2} + \big\|\nabla P_k\xi_i\big\|^2_{L^2} + \int_\tau^1 \frac{1}{\tau'}\big\|P_k\nabla_\tau\xi_i\big\|^2_{L^2}d\tau' \lesssim \Big(\big\|\nabla P_k\xi_i\big\|^2_{L^2} + \big\|P_k\nabla_\tau\xi_i\big\|^2_{L^2}\Big)\Big|_{\tau=1}+$$

$$+\int_\tau^1 (\tau')^2\big\|\nabla[P_k,\nabla_4]\xi_i\big\|^2_{L^2}d\tau' + \int_\tau^1 (\tau')^2\big\|P_k\xi_i\big\|^2_{L^2}d\tau' + \sum_{j=1}^{I}\int_\tau^1 \big\|[P_k,\psi]\nabla\xi_j\big\|^2_{L^2}d\tau'+$$

$$+\sum_{j=1}^{I}\int_\tau^1 \big\|P_k\nabla\xi_j\big\|^2_{L^2}d\tau' + \int_\tau^1 \tau'\big\|P_kF^i_m\big\|^2_{L^2}d\tau' + \int_\tau^1 (\tau')^3\big\|[\nabla_4,P_k]\nabla_\tau\xi_i\big\|^2_{L^2}d\tau'.$$

We use Lemma 6.3 and Lemma 6.4 to get:

$$\big\|P_k\nabla_\tau\xi_i\big\|^2_{L^2} + \big\|\nabla P_k\xi_i\big\|^2_{L^2} + \int_\tau^1 \frac{1}{\tau'}\big\|P_k\nabla_\tau\xi_i\big\|^2_{L^2}d\tau' \lesssim$$

$$\lesssim \Big(\big\|\nabla P_k\xi_i\big\|^2_{L^2} + \big\|P_k\nabla_\tau\xi_i\big\|^2_{L^2}\Big)\Big|_{\tau=1} + \int_\tau^1 (\tau')^2\big\|\underline{\widetilde{P}}_k\nabla\xi_i\big\|^2_{L^2}d\tau' + \int_\tau^1 (\tau')^2\big\|P_k\xi_i\big\|^2_{L^2}d\tau'+$$

$$+\sum_{j=1}^{I}\int_\tau^1 2^{-2k}\big\|\xi_j\big\|^2_{H^1}d\tau' + \sum_{j=1}^{I}\int_\tau^1 \big\|P_k\nabla\xi_j\big\|^2_{L^2}d\tau' + \int_\tau^1 \tau'\big\|P_kF^i_m\big\|^2_{L^2}d\tau'+$$

$$+\int_\tau^1 (\tau')^3\big\|\underline{\widetilde{P}}_k\nabla_\tau\xi_i\big\|^2_{L^2}d\tau' + \int_\tau^1 2^{-2k}(\tau')^3\big\|\nabla_\tau\xi_i\big\|^2_{L^2}d\tau'.$$

We multiply each inequality by 2^k and sum for all $1 \leq i \leq I$:

$$\sum_{i=1}^{I} 2^k \big\|P_k \nabla_\tau \xi_i\big\|_{L^2}^2 + \sum_{i=1}^{I} 2^k \big\|\nabla P_k \xi_i\big\|_{L^2}^2 + \sum_{i=1}^{I} \int_\tau^1 \frac{2^k}{\tau'} \big\|P_k \nabla_\tau \xi_i\big\|_{L^2}^2 d\tau' \lesssim$$

$$\lesssim \sum_{i=1}^{I} 2^k \Big(\big\|\nabla P_k \xi_i\big\|_{L^2}^2 + \big\|P_k \nabla_\tau \xi_i\big\|_{L^2}^2 \Big)\Big|_{\tau=1} +$$

$$+ \sum_{i=1}^{I} 2^k \int_\tau^1 (\tau')^2 \big\|\underline{\widetilde{P}}_k \nabla \xi_i\big\|_{L^2}^2 d\tau' + \sum_{i=1}^{I} 2^k \int_\tau^1 (\tau')^2 \big\|P_k \xi_i\big\|_{L^2}^2 d\tau' + \sum_{i=1}^{I} \int_\tau^1 2^{-k} \big\|\xi_i\big\|_{H^1}^2 d\tau' +$$

$$+ \sum_{i=1}^{I} 2^k \int_\tau^1 \tau' \big\|P_k F_m^i\big\|_{L^2}^2 d\tau' + \sum_{i=1}^{I} 2^k \int_\tau^1 (\tau')^3 \big\|\underline{\widetilde{P}}_k \nabla_\tau \xi_i\big\|_{L^2}^2 d\tau' + \sum_{i=1}^{I} \int_\tau^1 2^{-k} (\tau')^3 \big\|\nabla_\tau \xi_i\big\|_{L^2}^2 d\tau'.$$

We then sum for all $k \geq 0$, using the preliminary estimate as well to obtain:

$$\sum_{i=1}^{I} \big\|\nabla_\tau \xi_i\big\|_{H^{1/2}}^2 + \sum_{i=1}^{I} \big\|\xi_i\big\|_{H^{3/2}}^2 + \sum_{i=1}^{I} \int_\tau^1 \frac{1}{\tau'} \big\|\nabla_\tau \xi_i\big\|_{H^{1/2}}^2 d\tau' \lesssim \sum_{i=1}^{I} \Big(\big\|\xi_i\big\|_{H^{3/2}}^2 + \big\|\nabla_\tau \xi_i\big\|_{H^{1/2}}^2 \Big)\Big|_{\tau=1} +$$

$$+ \sum_{i=1}^{I} \int_\tau^1 \big\|\xi_i\big\|_{H^{3/2}}^2 d\tau' + \sum_{i=1}^{I} \int_\tau^1 \tau' \big\|F_m^i\big\|_{H^{1/2}}^2 d\tau' + \sum_{i=1}^{I} \int_\tau^1 (\tau')^3 \big\|\nabla_\tau \xi_i\big\|_{H^{1/2}}^2 d\tau'.$$

We use Grönwall and we bound the initial data term at $\tau = 1$ as before (using C_2) to obtain the conclusion. □

8.2 Preliminary Estimates for the Singular Quantities

For every $0 \leq m \leq M$, we write the equation (5.14) for $\nabla^m \Phi_0$ as:

$$\nabla_\tau \big(\nabla_\tau \nabla^m \Phi_0 \big) + \frac{1}{\tau} \nabla_\tau \nabla^m \Phi_0 - 4 \Delta \nabla^m \Phi_0 = \psi \nabla^{m+1} \Phi_0 + F_m', \tag{8.6}$$

where we denote $F_m' = \psi \sum_{i=1}^{I} \nabla^{m+1} \Phi_i + F_m^0$. We prove the preliminary estimate (1.46) as outlined in Section 1.3.3 of the Introduction:

Proposition 8.2. *For all $0 \leq m \leq M$ and $\tau \in [0,1]$ we have the estimates:*

$$\big\|\tau \nabla_\tau \nabla^m \Phi_0\big\|_{L^2}^2 + \big\|\tau \nabla^{m+1} \Phi_0\big\|_{L^2}^2 + \tau \big\|\nabla^m \Phi_0\big\|_{L^2}^2 + \int_\tau^1 \tau' \big\|\nabla^{m+1} \Phi_0\big\|_{L^2}^2 d\tau' \lesssim$$

$$\lesssim \Big(\big\|\Phi_0\big\|_{H^{m+3/2}}^2 + \big\|\nabla_\tau \Phi_0\big\|_{H^{m+1/2}}^2 \Big)\Big|_{\tau=1} + \int_\tau^1 (\tau')^2 \big\|F_m'\big\|_{L^2}^2 d\tau'.$$

Proof. For the purpose of this proof, we denote $\xi_m = \nabla^m \Phi_0$. We can rewrite equation (8.6) as:

$$\nabla_\tau \big(\tau \nabla_\tau \xi_m \big) - 4 \tau \Delta \xi_m = \tau \psi \nabla \xi_m + \tau F_m'.$$

Using Lemma 7.1, we obtain the energy estimate:

$$\big\|\tau\nabla_\tau\xi_m\big\|_{L^2}^2+\big\|\tau\nabla\xi_m\big\|_{L^2}^2+\int_\tau^1\tau'\big\|\nabla\xi_m\big\|_{L^2}^2d\tau'\lesssim\Big(\big\|\xi_m\big\|_{H^1}^2+\big\|\nabla_\tau\xi_m\big\|_{L^2}^2\Big)\Big|_{\tau=1}$$

$$+\int_\tau^1\tau'\big\|\nabla\xi_m\big\|_{L^2}\Big(\tau'\big\|\nabla_\tau\xi_m\big\|_{L^2}+(\tau')^2\big\|[\nabla,\nabla_4]\xi_m\big\|_{L^2}\Big)d\tau'+\int_\tau^1(\tau')^2\big\|F_m'\big\|_{L^2}^2d\tau'.$$

Using Grönwall, we obtain:

$$\big\|\tau\nabla_\tau\xi_m\big\|_{L^2}^2+\big\|\tau\nabla\xi_m\big\|_{L^2}^2+\int_\tau^1\tau'\big\|\nabla\xi_m\big\|_{L^2}^2d\tau'\lesssim$$

$$\Big(\big\|\xi_m\big\|_{H^1}^2+\big\|\nabla_\tau\xi_m\big\|_{L^2}^2\Big)\Big|_{\tau=1}+\int_\tau^1(\tau')^4\big\|\xi_m\big\|_{L^2}^2+\int_\tau^1(\tau')^2\big\|F_m'\big\|_{L^2}^2.$$

Similarly, we have the estimate:

$$\big\|\xi_m\big\|_{L^2}^2\lesssim\big\|\xi_m\big\|_{L^2}^2\Big|_{\tau=1}+\int_\tau^1\big\|\xi_m\big\|_{L^2}\big\|\nabla_\tau\xi_m\big\|_{L^2}d\tau'.$$

In particular, this implies:

$$\tau^4\big\|\xi_m\big\|_{L^2}^2\lesssim\big\|\xi_m\big\|_{L^2}^2\Big|_{\tau=1}+\int_\tau^1(\tau')^4\big\|\nabla_\tau\xi_m\big\|_{L^2}^2d\tau',$$

which gives in the above inequality:

$$\big\|\tau\nabla_\tau\xi_m\big\|_{L^2}^2+\big\|\tau\nabla\xi_m\big\|_{L^2}^2+\int_\tau^1\tau'\big\|\nabla\xi_m\big\|_{L^2}^2d\tau'\lesssim\Big(\big\|\xi_m\big\|_{H^1}^2+\big\|\nabla_\tau\xi_m\big\|_{L^2}^2\Big)\Big|_{\tau=1}+$$

$$+\int_\tau^1(\tau')^4\big\|\nabla_\tau\xi_m\big\|_{L^2}^2d\tau'+\int_\tau^1(\tau')^2\big\|F_m'\big\|_{L^2}^2d\tau'.$$

By Grönwall, we obtain:

$$\big\|\tau\nabla_\tau\xi_m\big\|_{L^2}^2+\big\|\tau\nabla\xi_m\big\|_{L^2}^2+\int_\tau^1\tau'\big\|\nabla\xi_m\big\|_{L^2}^2d\tau'\lesssim\Big(\big\|\xi_m\big\|_{H^1}^2+\big\|\nabla_\tau\xi_m\big\|_{L^2}^2\Big)\Big|_{\tau=1}+\int_\tau^1(\tau')^2\big\|F_m'\big\|_{L^2}^2.$$

Using this, we also have:

$$\tau\big\|\xi_m\big\|_{L^2}^2\lesssim\big\|\xi_m\big\|_{L^2}^2\Big|_{\tau=1}+\int_\tau^1\tau'\big\|\xi_m\big\|_{L^2}\cdot\big\|\nabla_\tau\xi_m\big\|_{L^2}$$

$$\lesssim\big\|\xi_m\big\|_{L^2}^2\Big|_{\tau=1}+\int_\tau^1\sqrt{\tau'}\big\|\xi_m\big\|_{L^2}^2+\int_\tau^1(\tau')^{3/2}\big\|\nabla_\tau\xi_m\big\|_{L^2}^2$$

$$\lesssim\Big(\big\|\xi_m\big\|_{H^1}^2+\big\|\nabla_\tau\xi_m\big\|_{L^2}^2\Big)\Big|_{\tau=1}+\int_\tau^1(\tau')^2\big\|F_m'\big\|_{L^2}^2d\tau'+\int_\tau^1\sqrt{\tau'}\big\|\xi_m\big\|_{L^2}^2d\tau'.$$

We obtain the conclusion using Grönwall and bounding the initial data term at $\tau=1$. □

8.3 Low Frequency Regime Estimates

For the rest of Section 8, we denote the top order term $\xi = \nabla^M \Phi_0$. We write (5.14) as:

$$\nabla_\tau\big(\nabla_\tau \xi\big) + \frac{1}{\tau}\nabla_\tau \xi - 4\Delta\xi = \psi\nabla\xi + F'_M, \tag{8.7}$$

where we denote $F'_M = \psi\sum_{i=1}^I \nabla^{M+1}\Phi_i + F^0_M$.

As outlined in Section 1.3.3 of the Introduction, we consider $X = 2^{x+1}$ to be a large constant (so $x > 0$ is large as well), to be chosen later depending only on M, C_0, C_2. We consider the following regimes:

- Negative frequencies $k < 0$;
- Low frequency regime $0 \leq k < x$ for all $\tau \in [0,1]$;
- Low frequency regime $k \geq x$ for $\tau \in [0, X2^{-k-1}]$;
- High frequency regime $k \geq x$ for $\tau \in [X2^{-k-1}, 1]$.

Notation. In addition to our previous notation convention, we write $A \lesssim_X B$ for some quantities $A, B > 0$ if there exists a constant $C > 0$ depending only on the constants M, C_0, C_2 and X, such that $A \leq CB$. Otherwise, if we write $A \lesssim B$ then the implicit constant C is independent of X.

In order to deal with the negative frequencies, we can simply use the preliminary estimate (1.46). Similarly, we have the following bound in the low frequency regime $0 \leq k < x$:

Proposition 8.3. *For $0 \leq k < x$, we have the low frequency regime estimate for all $\tau \in [0,1]$:*

$$\big\|\tau P_k \nabla_\tau \xi\big\|^2_{L^2} + \big\|\tau\nabla P_k \xi\big\|^2_{L^2} \lesssim_X 2^{-3k}\mathcal{D}_{II} + 2^{-3k}\int_\tau^1 (\tau')^2 \big\|F'_M\big\|^2_{L^2} d\tau'.$$

Proof. Since $0 \leq k < x$, we have using the preliminary estimate in Proposition 8.2:

$$\begin{aligned}\big\|\tau\nabla P_k \xi\big\|^2_{L^2} + \big\|\tau P_k \nabla_\tau \xi\big\|^2_{L^2} &\lesssim_X 2^{-3k}\Big(\big\|\tau\xi\big\|^2_{L^2} + \big\|\tau\nabla_\tau\xi\big\|^2_{L^2}\Big) \\ &\lesssim_X 2^{-3k}\mathcal{D}_{II} + 2^{-3k}\int_\tau^1 (\tau')^2 \big\|F'_M\big\|^2_{L^2} d\tau'.\end{aligned}$$

□

We prove the main estimate in the low frequency regime $k \geq x$ for $\tau \in [0, X2^{-k-1}]$. The idea is to prove a similar estimate to (1.46) for $P_k\xi$. We note that we follow the argument for the toy problem in [Cic24, Section 9], while keeping track of the inhomogeneous terms.

Proposition 8.4. *For any $0 \leq \tau < X2^{-k-1} \leq 1$ we have the low frequency regime estimate:*

$$\big\|\tau P_k \nabla_\tau \xi\big\|_{L^2}^2 + \big\|\tau \nabla P_k \xi\big\|_{L^2}^2 \lesssim X^2 2^{-2k}\Big(\big\|P_k \nabla_\tau \xi\big\|_{L^2}^2 + \big\|\nabla P_k \xi\big\|_{L^2}^2\Big)\Big|_{\tau = X2^{-k-1}} + C_X 2^{-3k}\mathcal{D}_{II} +$$
$$+ C_X 2^{-k}\int_\tau^{X2^{-k-1}} (\tau')^2 \big\|P_k F_M'\big\|_{L^2}^2 d\tau' + C_X 2^{-3k}\int_\tau^1 (\tau')^2\big\|F_M'\big\|_{L^2}^2 d\tau'.$$

Proof. We apply P_k to (8.7) to obtain for any $k \geq x$:

$$\nabla_\tau\big(\tau P_k \nabla_\tau \xi\big) - 4\tau \Delta P_k \xi = \tau P_k\big(\psi \nabla \xi\big) + \tau P_k F_M' + [\nabla_\tau, P_k]\tau \nabla_\tau \xi.$$

We contract the equation with $\tau P_k \nabla_\tau \xi$ and integrate by parts to obtain the following energy estimate:

$$\big\|\tau P_k \nabla_\tau \xi\big\|_{L^2}^2 + \big\|\tau \nabla P_k \xi\big\|_{L^2}^2 \lesssim$$
$$\lesssim X^2 2^{-2k}\Big(\big\|P_k \nabla_\tau \xi\big\|_{L^2}^2 + \big\|\nabla P_k \xi\big\|_{L^2}^2\Big)\Big|_{X2^{-k-1}} + \int_\tau^{X2^{-k-1}} (\tau')^3 \big\|\nabla P_k \xi\big\|_{L^2}\cdot \big\|\nabla [P_k, \nabla_4]\xi\big\|_{L^2} d\tau' +$$
$$+ \int_\tau^{X2^{-k-1}} (\tau')^3 \big\|\nabla P_k \xi\big\|_{L^2}\cdot \big\|[\nabla, \nabla_4]P_k \xi\big\|_{L^2} d\tau' + \int_\tau^{X2^{-k-1}} (\tau')^2 \big\|P_k \nabla_\tau \xi\big\|_{L^2}\cdot \big\|P_k\big(\psi \nabla \xi\big)\big\|_{L^2} d\tau' +$$
$$+ \int_\tau^{X2^{-k-1}} (\tau')^2 \big\|P_k \nabla_\tau \xi\big\|_{L^2}\cdot \big\|[P_k, \nabla_4]\tau' \nabla_\tau \xi\big\|_{L^2} d\tau' + \int_\tau^{X2^{-k-1}} (\tau')^2 \big\|P_k \nabla_\tau \xi\big\|_{L^2}\cdot \big\|P_k F_M'\big\|_{L^2} d\tau'.$$

We point out that similarly to the proof of Proposition 8.2, we obtain a bulk term with a favorable sign, which we drop. We use Lemma 6.3 to bound the commutation terms:

$$\big\|\tau P_k \nabla_\tau \xi\big\|_{L^2}^2 + \big\|\tau \nabla P_k \xi\big\|_{L^2}^2 \lesssim$$
$$\lesssim X^2 2^{-2k}\Big(\big\|P_k \nabla_\tau \xi\big\|_{L^2}^2 + \big\|\nabla P_k \xi\big\|_{L^2}^2\Big)\Big|_{X2^{-k-1}} + \int_\tau^{X2^{-k-1}} (\tau')^3 \big\|\nabla P_k \xi\big\|_{L^2}\big\|\xi\big\|_{H^1}$$
$$+ \int_\tau^{X2^{-k-1}} (\tau')^2 \big\|P_k \nabla_\tau \xi\big\|_{L^2}\big\|\nabla P_k \xi\big\|_{L^2} d\tau' + \int_\tau^{X2^{-k-1}} (\tau')^2 2^{-k}\big\|P_k \nabla_\tau \xi\big\|_{L^2}\big\|\xi\big\|_{H^1} d\tau' +$$
$$+ \int_\tau^{X2^{-k-1}} (\tau')^2 \big\|P_k \nabla_\tau \xi\big\|_{L^2}\big\|\tau' \nabla_\tau \xi\big\|_{L^2} d\tau' + \int_\tau^{X2^{-k-1}} (\tau')^2 \big\|P_k \nabla_\tau \xi\big\|_{L^2}\big\|P_k F_M'\big\|_{L^2} d\tau'.$$

Using Cauchy-Schwarz, we obtain the bound:

$$\big\|\tau P_k \nabla_\tau \xi\big\|_{L^2}^2 + \big\|\tau \nabla P_k \xi\big\|_{L^2}^2 \lesssim X^2 2^{-2k}\Big(\big\|P_k \nabla_\tau \xi\big\|_{L^2}^2 + \big\|\nabla P_k \xi\big\|_{L^2}^2\Big)\Big|_{X2^{-k-1}} +$$
$$+ \int_\tau^{X2^{-k-1}} \frac{2^k}{X}\big\|\tau' \nabla P_k \xi\big\|_{L^2}^2 d\tau' + \int_\tau^{X2^{-k-1}} \frac{2^k}{X}\big\|\tau' P_k \nabla_\tau \xi\big\|_{L^2}^2 d\tau' + C_X \int_\tau^{X2^{-k-1}} 2^{-3k}(\tau')^2\big\|\xi\big\|_{H^1}^2 +$$
$$+ C_X \int_\tau^{X2^{-k-1}} 2^{-k}(\tau')^4 \big\|\nabla_\tau \xi\big\|_{L^2}^2 d\tau' + 2^{-k} C_X \int_\tau^{X2^{-k-1}} (\tau')^2 \big\|P_k F_M'\big\|_{L^2}^2 d\tau'.$$

Using the Grönwall inequality for $\tau \leq X2^{-k-1} \leq 1$ we get:

$$\big\|\tau\nabla P_k\xi\big\|_{L^2}^2 + \big\|\tau P_k\nabla_\tau\xi\big\|_{L^2}^2 \lesssim$$

$$\lesssim X^2 2^{-2k}\Big(\big\|P_k\nabla_\tau\xi\big\|_{L^2}^2 + \big\|\nabla P_k\xi\big\|_{L^2}^2\Big)\Big|_{X2^{-k-1}} + C_X\int_\tau^{X2^{-k-1}} 2^{-3k}(\tau')^2\big\|\xi\big\|_{H^1}^2$$

$$+C_X\int_\tau^{X2^{-k-1}} 2^{-k}(\tau')^4\big\|\nabla_\tau\xi\big\|_{L^2}^2 + 2^{-k}C_X\int_\tau^{X2^{-k-1}} (\tau')^2\big\|P_kF'_M\big\|_{L^2}^2.$$

Finally, we use the above estimate to complete the proof. □

8.4 High Frequency Regime Estimates

In this section, we prove a high frequency regime estimate for the top order term $\xi = \nabla^M\Phi_0$. The proof is similar to that of Proposition 7.10, but in the current case the bulk term has an unfavorable sign, which creates several complications in Section 8.5. Moreover, it is essential that the implicit constant in the estimate obtained is independent of the parameter X.

Proposition 8.5. *ξ satisfies the high frequency regime estimate for any $\tau \in \big[X2^{-k-1}, 1\big]$:*

$$2^k\tau\big\|P_k\nabla_\tau\xi\big\|_{L^2}^2 + \frac{2^k}{\tau}\big\|P_k\xi\big\|_{L^2}^2 + 2^k\tau\big\|\nabla P_k\xi\big\|_{L^2}^2 \lesssim$$

$$\lesssim \left(2^k\big\|P_k\nabla_\tau\xi\big\|_{L^2}^2 + 2^k\big\|P_k\xi\big\|_{L^2}^2 + 2^k\big\|\nabla P_k\xi\big\|_{L^2}^2\right)\bigg|_{\tau=1}$$

$$+\int_\tau^1 \frac{2^k}{(\tau')^2}\big\|P_k\xi\big\|_{L^2}^2 + \int_\tau^1 2^k\tau'\big\|\underline{\widetilde{P}}_k\xi\big\|_{L^2}^2 + \int_\tau^1 2^k(\tau')^3\big\|\underline{\widetilde{P}}_k\nabla\xi\big\|_{L^2}^2 + \int_\tau^1 2^k(\tau')^3\big\|\underline{\widetilde{P}}_k\nabla_\tau\xi\big\|_{L^2}^2 +$$

$$+\int_\tau^1 \frac{\tau'}{2^k}\big\|\xi\big\|_{H^1}^2 d\tau' + \int_\tau^1 \frac{(\tau')^3}{2^k}\big\|\nabla_\tau\xi\big\|_{L^2}^2 d\tau' + \int_\tau^1 2^k\tau'\big\|P_kF'_M\big\|_{L^2}^2 d\tau'.$$

Proof. We introduce the new time variable $t = X^{-1}2^k\tau \in [1/2, X^{-1}2^k]$. The equation (8.7) becomes:

$$\nabla_t\big(\nabla_t\xi\big) + \frac{1}{t}\nabla_t\xi - 4\frac{X^2}{2^{2k}}\cdot\Delta\xi = \frac{X^2}{2^{2k}}\cdot\psi\nabla\xi + \frac{X^2}{2^{2k}}\cdot F'_M.$$

We multiply by $\sqrt{t}$ to get:

$$\nabla_t\big(\nabla_t(\xi\sqrt{t})\big) + \frac{1}{4t^2}\xi\sqrt{t} - 4\frac{X^2}{2^{2k}}\cdot\Delta\xi\sqrt{t} = \frac{X^2}{2^{2k}}\cdot\psi\nabla\xi\sqrt{t} + \frac{X^2}{2^{2k}}\cdot F'_M\sqrt{t}.$$

We apply P_k to the equation:

$$\nabla_t\big(P_k\nabla_t(\xi\sqrt{t})\big) + \frac{1}{4t^2}P_k\xi\sqrt{t} - 4\frac{X^2}{2^{2k}}\Delta P_k\xi\sqrt{t} = \frac{X^2}{2^{2k}}\psi\nabla P_k\xi\sqrt{t} + \frac{X^2}{2^{2k}}\psi[P_k,\nabla]\xi\sqrt{t}+$$

$$+\frac{X^2}{2^{2k}}[P_k,\psi]\nabla\xi\sqrt{t}+\frac{X^2P_kF_M'\sqrt{t}}{2^{2k}}+[\nabla_t,P_k]\nabla_t\xi\sqrt{t}.$$

We contract the equation with $P_k\nabla_t(\xi\sqrt{t})$ and integrate by parts. We notice that using the analogue of Lemma 7.1 for the new time variable t does not introduce any constants that depend on X. We obtain the energy estimate:

$$\big\|P_k\nabla_t\xi\sqrt{t}\big\|_{L^2}^2+\frac{1}{t^2}\big\|P_k\xi\sqrt{t}\big\|_{L^2}^2+\frac{X^2}{2^{2k}}\big\|\nabla P_k\xi\sqrt{t}\big\|_{L^2}^2\lesssim$$

$$\lesssim\left(X^{-1}2^k\big\|P_k\nabla_t\xi\big\|_{L^2}^2+\frac{X}{2^k}\big\|P_k\xi\big\|_{L^2}^2+\frac{X}{2^k}\big\|\nabla P_k\xi\big\|_{L^2}^2\right)\bigg|_{t=X^{-1}2^k}+\int_t^{X^{-1}2^k}\frac{1}{(t')^2}\big\|P_k\xi\big\|_{L^2}^2dt'$$

$$+\int_t^{X^{-1}2^k}\int_{S^n}\frac{1}{(t')^2}\big|P_k\xi\sqrt{t}\big|\cdot\big|[P_k,\nabla_t]\xi\sqrt{t'}\big|dt'+\int_t^{X^{-1}2^k}\int_{S^n}\frac{X^2}{2^{2k}}\big|\nabla P_k\xi\sqrt{t'}\big|\cdot\big|\nabla[P_k,\nabla_t]\xi\sqrt{t'}\big|dt'$$

$$+\int_t^{X^{-1}2^k}\int_{S^n}\frac{X^2}{2^{2k}}\big|\nabla P_k\xi\sqrt{t'}\big|\cdot\big|[\nabla,\nabla_t]P_k\xi\sqrt{t'}\big|dt'+\int_t^{X^{-1}2^k}\int_{S^n}\frac{X^2}{2^{2k}}\big|P_k\nabla_t\xi\sqrt{t'}\big|\cdot\big|\nabla P_k\xi\sqrt{t'}\big|dt'$$

$$+\int_t^{X^{-1}2^k}\int_{S^n}\frac{X^2}{2^{2k}}\big|P_k\nabla_t\xi\sqrt{t'}\big|\cdot\big|[P_k,\nabla]\xi\sqrt{t'}\big|dt'+\int_t^{X^{-1}2^k}\int_{S^n}\frac{X^2}{2^{2k}}\big|P_k\nabla_t\xi\sqrt{t'}\big|\cdot\big|[P_k,\psi]\nabla\xi\sqrt{t'}\big|dt'$$

$$+\int_t^{X^{-1}2^k}\int_{S^n}\frac{X^2}{2^{2k}}\big|P_k\nabla_t\xi\sqrt{t'}\big|\cdot\big|P_kF_M'\sqrt{t'}\big|dt'+\int_t^{X^{-1}2^k}\int_{S^n}\big|P_k\nabla_t\xi\sqrt{t'}\big|\cdot\big|[\nabla_t,P_k]\nabla_t\xi\sqrt{t'}\big|dt'.$$

We use Lemma 6.4, and Grönwall for $t\in\big[1/2,X^{-1}2^k\big]$ (to deal with the fifth and sixth error terms):

$$\big\|P_k\nabla_t\xi\sqrt{t}\big\|_{L^2}^2+\frac{1}{t^2}\big\|P_k\xi\sqrt{t}\big\|_{L^2}^2+\frac{X^2}{2^{2k}}\big\|\nabla P_k\xi\sqrt{t}\big\|_{L^2}^2\lesssim$$

$$\lesssim\left(X^{-1}2^k\big\|P_k\nabla_t\xi\big\|_{L^2}^2+\frac{X}{2^k}\big\|P_k\xi\big\|_{L^2}^2+\frac{X}{2^k}\big\|\nabla P_k\xi\big\|_{L^2}^2\right)\bigg|_{t=X^{-1}2^k}+\int_t^{X^{-1}2^k}\frac{1}{(t')^2}\big\|P_k\xi\big\|_{L^2}^2dt'+$$

$$+\int_t^{X^{-1}2^k}\frac{X^2}{2^{2k}t'}\big\|P_k\xi\sqrt{t'}\big\|_{L^2}\cdot\Big(\big\|\underline{P}_k\xi\sqrt{t'}\big\|_{L^2}+2^{-k}\big\|\xi\sqrt{t'}\big\|_{L^2}\Big)dt'+$$

$$+\int_t^{X^{-1}2^k}\frac{X^4t'}{2^{4k}}\big\|\nabla P_k\xi\sqrt{t'}\big\|_{L^2}\cdot\left(\big\|\underline{\widetilde{P}}_k\nabla\xi\sqrt{t'}\big\|_{L^2}+2^{-k}\big\|\xi\sqrt{t'}\big\|_{H^1}\right)$$

$$+\int_t^{X^{-1}2^k}\frac{X^2}{2^{3k}}\big\|P_k\nabla_t\xi\sqrt{t'}\big\|_{L^2}\cdot\big\|\xi\sqrt{t'}\big\|_{L^2}$$

$$+\int_t^{X^{-1}2^k}\frac{X^2}{2^{3k}}\big\|P_k\nabla_t\xi\sqrt{t'}\big\|_{L^2}\cdot\big\|\nabla\xi\sqrt{t'}\big\|_{L^2}dt'+\int_t^{X^{-1}2^k}\frac{X^2}{2^{2k}}\big\|P_k\nabla_t\xi\sqrt{t'}\big\|_{L^2}\big\|P_kF_M'\sqrt{t'}\big\|_{L^2}dt'+$$

$$+\int_t^{X^{-1}2^k}\frac{X^2t'}{2^{2k}}\big\|P_k\nabla_t\xi\sqrt{t'}\big\|_{L^2}\Big(\big\|\underline{\widetilde{P}}_k\nabla_t\xi\sqrt{t'}\big\|_{L^2}+2^{-k}\big\|\nabla_t\xi\sqrt{t'}\big\|_{L^2}\Big)dt'.$$

We use Grönwall for $t \in \left[1/2, X^{-1}2^k\right]$ to get:

$$\big\|P_k \nabla_t \xi \sqrt{t}\big\|_{L^2}^2 + \frac{1}{t^2}\big\|P_k \xi \sqrt{t}\big\|_{L^2}^2 + \frac{X^2}{2^{2k}}\big\|\nabla P_k \xi \sqrt{t}\big\|_{L^2}^2 \lesssim$$

$$\lesssim \left(X^{-1}2^k\big\|P_k \nabla_t \xi\big\|_{L^2}^2 + \frac{X}{2^k}\big\|P_k \xi\big\|_{L^2}^2 + \frac{X}{2^k}\big\|\nabla P_k \xi\big\|_{L^2}^2\right)\Big|_{t=X^{-1}2^k} + \int_t^{X^{-1}2^k} \frac{1}{(t')^2}\big\|P_k \xi\big\|_{L^2}^2 dt' +$$

$$+ \int_t^{X^{-1}2^k} \frac{X^3}{2^{3k}}\big\|\underline{\widetilde{P}}_k \xi \sqrt{t'}\big\|_{L^2}^2 + \int_t^{X^{-1}2^k} \frac{X^5 (t')^2}{2^{5k}}\big\|\underline{\widetilde{P}}_k \nabla \xi \sqrt{t'}\big\|_{L^2}^2 + \int_t^{X^{-1}2^k} \frac{X^3 (t')^2}{2^{3k}}\big\|\underline{\widetilde{P}}_k \nabla_t \xi \sqrt{t'}\big\|_{L^2}^2 +$$

$$+ \int_t^{X^{-1}2^k} \frac{X^3 t'}{2^{5k}}\big\|\xi\big\|_{H^1}^2 dt' + \int_t^{X^{-1}2^k} \frac{X^3 (t')^3}{2^{5k}}\big\|\nabla_t \xi\big\|_{L^2}^2 dt' + \int_t^{X^{-1}2^k} \frac{X^3 t'}{2^{3k}}\big\|P_k F'_M\big\|_{L^2}^2 dt'.$$

We change variables to τ and we obtain the conclusion. □

8.5 Improved High Frequency Estimates

The goal of this section is to improve the high frequency regime estimate of Proposition 8.5 in order to prove estimates in the high frequency regime only in terms of data and the inhomogeneous terms.

Proposition 8.6. *We denote by $\mathbf{1}_{k,\tau}$ the characteristic function of $\{1 \geq \tau \geq X2^{-k-1}\}$, and we denote:*

$$a_k(\tau) := \tau\big\|P_k \nabla_\tau \xi\big\|_{L^2}^2 + \frac{1}{\tau}\big\|P_k \xi\big\|_{L^2}^2 + \tau\big\|\nabla P_k \xi\big\|_{L^2}^2.$$

We have the improved high frequency estimate for any $\tau \in (0, 1]$:

$$\sum_{\tau \geq X2^{-k-1}} 2^k a_k(\tau)\mathbf{1}_{k,\tau} \lesssim \mathcal{D}_{II} + \int_\tau^1 \tau'\big\|F'_M\big\|_{H^{1/2}}^2 d\tau'. \tag{8.8}$$

The rest of this section is dedicated to the proof of this proposition. We follow the detailed outline of the proof from Section 1.3.3 of the Introduction, and we divide the proof into multiple steps. We note that the proof is very similar to the toy problem considered in [Cic24, Section 9], but in the current case we also need to keep track of the inhomogeneous terms.

Throughout the proof we use the schematic notation $\{d_k\}_{k\geq 0}$ for data terms at $\tau = 1$ which satisfy:

$$\sum_{k\geq 0} d_k \lesssim \mathcal{D}_{II}.$$

Consequences of Proposition 8.5. The starting point is the preliminary high frequency regime estimate in Proposition 8.5. Using the above notation, we have for all $\tau \in [X2^{-k-1}, 1]$:

$$2^k a_k(\tau)\mathbf{1}_{k,\tau} \lesssim 2^k a_k(1) + \mathbf{1}_{k,\tau}\int_\tau^1 \frac{2^k}{(\tau')^2}\big\|P_k\xi\big\|_{L^2}^2 d\tau' +$$

$$+\mathbf{1}_{k,\tau}\int_\tau^1 e_k(\tau') + \mathbf{1}_{k,\tau}\int_\tau^1 \overline{e_k}(\tau') + \int_\tau^1 2^k\tau'\big\|P_kF_M'\big\|_{L^2}^2,$$

where we define the energies:

$$e_k = 2^k\tau^3\big\|\underline{\widetilde{P}}_k\nabla\xi\big\|_{L^2}^2 + 2^k\tau^3\big\|\underline{\widetilde{P}}_k\nabla_\tau\xi\big\|_{L^2}^2,$$

$$\overline{e_k} = 2^k\tau\big\|\underline{\widetilde{P}}_k\xi\big\|_{L^2}^2 + \frac{\tau}{2^k}\big\|\xi\big\|_{H^1}^2 + \frac{\tau^3}{2^k}\big\|\nabla_\tau\xi\big\|_{L^2}^2.$$

The error term containing e_k is similar to the error terms with different projection operators of Section 7, as explained in Section 1.3.3. We deal with these terms towards the end of our argument, when summing the estimates obtained for all $k \geq x$. In the meantime we simply keep track of these terms, similarly to the inhomogeneous terms.

On the other hand, we can already bound the error terms containing $\overline{e_k}$ using the preliminary estimate in Proposition 8.2. We can bound the first term in $\overline{e_k}$ by the second term, using the finite band property for LP projections. We then apply (1.46) to get:

$$\int_\tau^1 \overline{e_k}(\tau')d\tau' \lesssim_X d_k + 2^{-k}\int_\tau^1 (\tau')^2\big\|F_M'\big\|_{L^2}^2 d\tau'.$$

As a consequence, we proved that for all $\tau \in [X2^{-k-1}, 1]$ we have the high frequency regime estimate:

$$2^k a_k(\tau)\mathbf{1}_{k,\tau} \lesssim C_X d_k + \mathbf{1}_{k,\tau}\int_\tau^1 \frac{2^k}{(\tau')^2}\big\|P_k\xi\big\|_{L^2}^2 +$$

$$+\mathbf{1}_{k,\tau}\int_\tau^1 e_k(\tau') + C_X 2^{-k}\int_\tau^1 (\tau')^2\big\|F_M'\big\|_{L^2}^2 + \int_\tau^1 2^k\tau'\big\|P_kF_M'\big\|_{L^2}^2.$$

Applying the refined Poincaré inequality. According to Section 1.3.3 of the Introduction, the second error term in the above estimate causes significant challenges. We also explained in Section 1.3.3 that we could not bound this term directly using the finite band property for LP projections. Instead, we use the refined Poincaré inequality for LP projections (6.28). As a result, we have for all $\tau \in [X2^{-k-1}, 1]$ and $\delta > 0$:

$$\int_\tau^1 \frac{2^k}{(\tau')^2}\big\|P_k\xi\big\|_{L^2}^2 d\tau' \lesssim \frac{1}{\delta}\int_\tau^1 \frac{2^{-k}}{(\tau')^2}\big\|\nabla P_k\xi\big\|_{L^2}^2 d\tau' +$$

$$+\delta\int_{\tau}^{1}\frac{1}{(\tau')^2}\sum_{l=0}^{k-1}2^{-8k+7l}\big\|\nabla P_l\xi\big\|_{L^2}^2d\tau'+\frac{1}{\delta}\int_{\tau}^{1}\frac{2^{-3k}}{(\tau')^2}\big\|\xi\big\|_{L^2}^2d\tau'.$$

The last term in this inequality is bounded using the preliminary estimate (1.46):

$$\int_{\tau}^{1}\frac{2^{-3k}}{(\tau')^2}\big\|\xi\big\|_{L^2}^2d\tau'\lesssim_X 2^{-k}\mathcal{D}_{II}+2^{-k}\int_{\tau}^{1}(\tau')^2\big\|F_M'\big\|_{L^2}^2d\tau'.$$

As a result, there exist constants $C', C_X, C_{X,\delta}>0$ such that for all $k\geq x$ and $\tau\in[X2^{-k-1},1]$:

$$2^ka_k(\tau)\leq C_{X,\delta}d_k+\frac{C'}{\delta}\int_{\tau}^{1}\frac{2^{-2k}}{(\tau')^3}\cdot 2^ka_k(\tau')d\tau'+C'\delta\int_{\tau}^{1}\frac{1}{(\tau')^2}\sum_{l=0}^{k-1}2^{-8k+7l}\big\|\nabla P_l\xi\big\|_{L^2}^2d\tau'+$$

$$+C_{X,\delta}\int_{\tau}^{1}e_k(\tau')d\tau'+C_{X,\delta}2^{-k}\int_{\tau}^{1}(\tau')^2\big\|F_M'\big\|_{L^2}^2d\tau'+C_{X,\delta}\int_{\tau}^{1}2^k\tau'\big\|P_kF_M'\big\|_{L^2}^2d\tau'.$$

We can deal with the second error term by using Grönwall for $\tau\in[X2^{-k-1},1]$. We compute:

$$\exp\left(\frac{C'}{\delta}\int_{\tau}^{1}\frac{2^{-2k}}{(\tau')^3}d\tau'\right)\leq\exp\left(\frac{C'}{\delta}\cdot\frac{1}{X^2}\right)\leq 2,$$

where we fix $X>0$ large enough, depending on C' and δ, such that:

$$\frac{C'}{\delta}\cdot\frac{1}{X^2}\leq\log(2). \tag{8.9}$$

Therefore, we have for all $\tau\in[X2^{-k-1},1]$:

$$2^ka_k(\tau)\leq C_\delta d_k+2C'\delta\int_{\tau}^{1}\frac{1}{(\tau')^2}\sum_{l=0}^{k-1}2^{-8k+7l}\big\|\nabla P_l\xi\big\|_{L^2}^2+$$

$$+C_\delta\int_{\tau}^{1}e_k(\tau')+C_\delta\int_{\tau}^{1}2^{-k}(\tau')^2\big\|F_M'\big\|_{L^2}^2+2^k\tau'\big\|P_kF_M'\big\|_{L^2}^2.$$

Bounding the low frequency regime error terms. As a consequence of the refined Poincaré inequality, the second term in the above has both a low frequency regime and a high frequency regime part. We want to separate these two, and apply the low frequency regime estimates in Section 8.3. This process creates a sum of discrete error terms which pose additional difficulties. We notice that we can write:

$$\mathbf{1}_{k,\tau}\int_{\tau}^{1}\frac{1}{(\tau')^3}\sum_{l=0}^{k-1}2^{-8k+7l}\tau'\big\|\nabla P_l\xi\big\|_{L^2}^2d\tau'\leq\mathbf{1}_{k,\tau}\int_{\tau}^{1}\frac{1}{(\tau')^3}\sum_{l=x}^{k-1}2^{-8k+6l}\cdot 2^la_l(\tau')\mathbf{1}_{l,\tau'}d\tau'+$$

$$+\mathbf{1}_{k,\tau}\int_{\tau}^{1}\frac{1}{(\tau')^4}\sum_{l=0}^{x-1}2^{-8k+7l}\big\|\tau'\nabla P_l\xi\big\|_{L^2}^2d\tau'+\mathbf{1}_{k,\tau}\int_{\tau}^{1}\frac{1}{(\tau')^4}\sum_{\tau'<X2^{-l-1}\leq 1}2^{-8k+7l}\big\|\tau'\nabla P_l\xi\big\|_{L^2}^2d\tau',$$

where the first term is in the high frequency regime and the last two terms are in the low frequency regime. We use Proposition 8.3 to obtain:

$$\mathbf{1}_{k,\tau}\int_\tau^1 \frac{1}{(\tau')^4}\sum_{l=0}^{x-1}2^{-8k+7l}\big\|\tau'\nabla P_l\xi\big\|_{L^2}^2 \lesssim_X \mathcal{D}_{II}\cdot\mathbf{1}_{k,\tau}\frac{2^{-8k}}{\tau^7}+\mathbf{1}_{k,\tau}\frac{2^{-8k}}{\tau^7}\int_\tau^1(\tau')^2\big\|F_M'\big\|_{L^2}^2$$

$$\lesssim_X d_k+2^{-k}\int_\tau^1(\tau')^2\big\|F_M'\big\|_{L^2}^2.$$

Similarly, we use Proposition 8.4 to obtain:

$$\mathbf{1}_{k,\tau}\int_\tau^1\frac{1}{(\tau')^4}\sum_{\tau'<X2^{-l-1}\le 1}2^{-8k+7l}\big\|\tau'\nabla P_l\xi\big\|_{L^2}^2d\tau'=$$

$$=\mathbf{1}_{k,\tau}\sum_{\tau<X2^{-l-1}\le 1}\int_\tau^{X2^{-l-1}}\frac{1}{(\tau')^4}2^{-8k+7l}\big\|\tau'\nabla P_l\xi\big\|_{L^2}^2d\tau'$$

$$\lesssim\mathbf{1}_{k,\tau}X\sum_{\tau<X2^{-l-1}\le 1}\frac{2^{-8k+6l}}{\tau^3}a_l(X2^{-l-1})+C_X\mathcal{D}_{II}\cdot\mathbf{1}_{k,\tau}\frac{2^{-8k}}{\tau^7}+$$

$$+C_X\mathbf{1}_{k,\tau}\frac{2^{-8k}}{\tau^7}\int_\tau^1(\tau')^2\big\|F_M'\big\|_{L^2}^2d\tau'+C_X\mathbf{1}_{k,\tau}\frac{2^{-8k}}{\tau^7}\int_\tau^1\tau'\big\|F_M'\big\|_{H^{1/2}}^2d\tau'.$$

We notice that in the above estimate the last term is obtained by bounding:

$$\sum_{\tau<X2^{-l-1}\le 1}\int_\tau^{X2^{-l-1}}\frac{1}{(\tau')^4}2^{-8k+6l}\int_{\tau'}^{X2^{-l-1}}\tilde{\tau}^2\big\|P_lF_M'\big\|_{L^2}^2d\tilde{\tau}d\tau'$$

$$\lesssim_X\sum_{\tau<X2^{-l-1}\le 1}\frac{1}{\tau^3}2^{-8k+4l}\int_\tau^1\tilde{\tau}2^l\big\|P_lF_M'\big\|_{L^2}^2d\tilde{\tau}.$$

As a result, we proved that there exist constants $C'', C_\delta>0$ such that for all $\tau\in[X2^{-k-1},1]$:

$$2^ka_k(\tau)\le C_\delta d_k+C''\delta\cdot\mathbf{1}_{k,\tau}\int_\tau^1\frac{1}{(\tau')^3}\sum_{l=x}^{k-1}2^{-8k+6l}\cdot 2^la_l(\tau')\mathbf{1}_{l,\tau'}d\tau'+$$

$$+C''\delta X\mathbf{1}_{k,\tau}\sum_{\tau<X2^{-l-1}\le 1}\frac{2^{-8k+6l}}{\tau^3}a_l(X2^{-l-1})+C_\delta\int_\tau^1e_k(\tau')d\tau'+C_\delta\int_\tau^1 2^k\tau'\big\|P_kF_M'\big\|_{L^2}^2d\tau'$$

$$+C_\delta 2^{-k}\int_\tau^1(\tau')^2\big\|F_M'\big\|_{L^2}^2d\tau'+C_\delta 2^{-k}\int_\tau^1\tau'\big\|F_M'\big\|_{H^{1/2}}^2d\tau'.$$

Grönwall-like inequality. Our next goal is to deal with the second term in the above estimate using a suitable Grönwall-like inequality. We set-up the problem in order to isolate this error term. We consider $\delta>0$ to be small enough, so that:

$$C''\delta<\frac{1}{10}. \tag{8.10}$$

We introduce some notation for the error terms on the right hand side of the above estimate:

$$\begin{aligned}
S_k(\tau) &= \mathbf{1}_{k,\tau} C'' \delta X \sum_{\tau < X2^{-l-1} \leq 1} \frac{2^{-8k+6l}}{\tau^3} a_l(X2^{-l-1}),\\
E_k(\tau) &= \mathbf{1}_{k,\tau} C_\delta \int_\tau^1 e_k(\tau')d\tau',\\
I_k(\tau) &= \mathbf{1}_{k,\tau} C_\delta 2^{-k} \int_\tau^1 \tau' \big\|F'_M\big\|^2_{H^{1/2}} d\tau' + \mathbf{1}_{k,\tau} C_\delta \int_\tau^1 2^k \tau' \big\|P_k F'_M\big\|^2_{L^2} d\tau',\\
A_k(\tau) &= C_\delta d_k + S_k(\tau) + E_k(\tau) + I_k(\tau).
\end{aligned}$$

Using this notation, we have that for all $k \geq x$ and $\tau \in [X2^{-k-1}, 1]$:

$$2^k a_k(\tau) \mathbf{1}_{k,\tau} \leq A_k(\tau) + \frac{1}{10} \cdot \mathbf{1}_{k,\tau} \int_\tau^1 \frac{1}{(\tau')^3} \sum_{l=x}^{k-1} 2^{-8k+6l} \cdot 2^l a_l(\tau') \mathbf{1}_{l,\tau'} d\tau'. \tag{8.11}$$

This motivates us to prove the following Grönwall-like lemma:

Lemma 8.1. *We consider the functions $u, A, b, c : \mathbb{N} \times [0,1] \to [0,\infty)$, which for all $k \geq x,\ \tau \in (0,1]$ satisfy the inequality:*

$$u(k,\tau) \leq A(k,\tau) + b(k) \int_\tau^1 \sum_{l=x}^{k-1} c(l,\tau') u(l,\tau') d\tau' \tag{8.12}$$

Then, we have that for all $k \geq x,\ \tau \in (0,1]$:

$$u(k,\tau) \leq A(k,\tau) + b(k) \int_\tau^1 \sum_{l=x}^{k-1} c(l,\tau') A(l,\tau') \prod_{j=l+1}^{k-1} \Big(1 + \int_\tau^{\tau'} b(j) c(j,\tau'') d\tau''\Big) d\tau'. \tag{8.13}$$

Proof. We prove this by induction on k. In the case $k = x$, we have from (8.12) that $u(x,\tau) \leq A(x,\tau)$ as desired. Next, we assume that (8.13) holds for $x, x+1, \ldots, k$ and we prove it for $k+1$. We have:

$$u(k+1,\tau) \leq A(k+1,\tau) + b(k+1) \int_\tau^1 \sum_{l=x}^{k} c(l,\tau') u(l,\tau') d\tau' \leq A(k+1,\tau) +$$

$$+ b(k+1) \int_\tau^1 \sum_{l=x}^{k} c(l,\tau') \Big\{ A(l,\tau') + b(l) \int_{\tau'}^1 \sum_{i=x}^{l-1} c(i,\tau'') A(i,\tau'') \prod_{j=i+1}^{l-1} \Big(1 + \int_{\tau'}^{\tau''} b(j) c(j,\tilde{\tau})\Big) d\tau'' \Big\}$$

$$\leq A(k+1,\tau) + b(k+1) \int_\tau^1 \sum_{l=x}^{k} c(l,\tau') A(l,\tau') d\tau' +$$

$$+ b(k+1) \int_\tau^1 \int_{\tau'}^1 \sum_{l=x}^{k} \sum_{i=x}^{l-1} \Big\{ c(l,\tau') b(l) c(i,\tau'') A(i,\tau'') \prod_{j=i+1}^{l-1} \Big(1 + \int_{\tau'}^{\tau''} b(j) c(j,\tilde{\tau}) d\tilde{\tau}\Big) \Big\} d\tau'' d\tau'$$

$$\leq A(k+1,\tau) + b(k+1)\int_{\tau}^{1}\sum_{l=x}^{k} c(l,\tau')A(l,\tau')d\tau' +$$

$$+b(k+1)\int_{\tau}^{1}\int_{\tau}^{\tau''}\sum_{i=x}^{k-1}\sum_{l=i+1}^{k}\Big\{c(l,\tau')b(l)c(i,\tau'')A(i,\tau'')\prod_{j=i+1}^{l-1}\Big(1+\int_{\tau'}^{\tau''} b(j)c(j,\tilde{\tau})d\tilde{\tau}\Big)\Big\}d\tau' d\tau''$$

$$\leq A(k+1,\tau) + b(k+1)\int_{\tau}^{1}\sum_{l=x}^{k} c(l,\tau')A(l,\tau')d\tau' +$$

$$+b(k+1)\int_{\tau}^{1}\sum_{i=x}^{k-1} c(i,\tau'')A(i,\tau'')\Big\{\int_{\tau}^{\tau''}\sum_{l=i+1}^{k} c(l,\tau')b(l)\prod_{j=i+1}^{l-1}\Big(1+\int_{\tau'}^{\tau''} b(j)c(j,\tilde{\tau})d\tilde{\tau}\Big)d\tau'\Big\}d\tau''$$

$$\leq A(k+1,\tau) + b(k+1)\int_{\tau}^{1}\sum_{l=x}^{k} c(l,\tau')A(l,\tau')d\tau' +$$

$$+b(k+1)\int_{\tau}^{1}\sum_{i=x}^{k} c(i,\tau'')A(i,\tau'')\Big\{\int_{\tau}^{\tau''}\sum_{l=i+1}^{k} c(l,\tau')b(l)\prod_{j=i+1}^{l-1}\Big(1+\int_{\tau}^{\tau''} b(j)c(j,\tilde{\tau})d\tilde{\tau}\Big)d\tau'\Big\}d\tau''$$

We conclude, since we have:

$$1+\sum_{l=i+1}^{k}\int_{\tau}^{\tau''} c(l,\tau')b(l)d\tau'\cdot\prod_{j=i+1}^{l-1}\Big(1+\int_{\tau}^{\tau''} b(j)c(j,\tilde{\tau})d\tilde{\tau}\Big) = \prod_{j=i+1}^{k}\Big(1+\int_{\tau}^{\tau''} b(j)c(j,\tilde{\tau})d\tilde{\tau}\Big).$$

□

Consequences of Lemma 8.1. We use the Grönwall lemma established above for the inequality (8.11).

Corollary 8.1. *For all $k\geq x$ and $\tau\in[X2^{-k-1},1]$, we have the high frequency regime estimate:*

$$2^k a_k(\tau)\mathbf{1}_{k,\tau} \lesssim A_k(\tau) + 2^{-7k}\int_{\tau}^{1}\sum_{l=0}^{k-1}\frac{2^{5l}}{(\tau')^3}\cdot A_l(\tau')\mathbf{1}_{l,\tau'}d\tau' \tag{8.14}$$

Proof. We apply Lemma 8.1 for the inequality (8.11), where:

$$u(k,\tau)=2^k a_k(\tau)\mathbf{1}_{k,\tau},\ b(k)=\frac{1}{10}\cdot 2^{-8k},\ c(k,\tau)=\tau^{-3}2^{6k}\mathbf{1}_{k,\tau}.$$

From (8.13) we obtain for $\tau\in[X2^{-k-1},1]$:

$$2^k a_k(\tau)\mathbf{1}_{k,\tau}\leq A_k(\tau)+2^{-8k}\int_{\tau}^{1}\sum_{l=x}^{k-1}A_l(\tau')\cdot\frac{1}{(\tau')^3}2^{6l}\mathbf{1}_{l,\tau'}\prod_{j=l+1}^{k-1}\Big(1+\int_{\tau}^{\tau'} b(j)c(j,\tau'')d\tau''\Big)d\tau'.$$

In order to bound the above, we first note that:

$$\int_{\tau}^{\tau'} b(j)c(j,\tau'')d\tau''=\frac{1}{10}\cdot 2^{-2j}\int_{\tau}^{\tau'}\frac{\mathbf{1}_{j,\tau''}}{(\tau'')^3}d\tau''\leq\min\big(X^{-2},2^{-2j}\tau^{-2}\big)\leq\min\big(1,2^{-2j}\tau^{-2}\big).$$

In this bound we used the good control of the constant in the definition of $b(k)$, obtained using the smallness of δ in (8.10). As a result, we obtain for all $\tau \in [X2^{-k-1}, 1]$:

$$\prod_{j=l+1}^{k-1} \left(1 + \int_{\tau}^{\tau'} b(j)c(j,\tau'')d\tau''\right) \leq \prod_{j=l+1}^{k-1} \left(\min\left(2, 1+2^{-2j}\tau^{-2}\right)\right) \lesssim 1 + 2^{-l}\tau^{-1},$$

where we bounded the terms with $2^{-j} < \tau$ using the inequality $x + 1 \leq e^x$. Finally, we conclude by noticing that:

$$2^{-8k}2^{6l} \cdot 2^{-l}\tau^{-1} \leq 2^{-7k}2^{5l} \cdot \frac{2^{-k}}{\tau} \leq 2^{-7k}2^{5l} \cdot \frac{2}{X} \leq 2^{-7k}2^{5l}.$$

□

We use the definition of $A_k(\tau)$ to compute the new error terms obtained on the RHS of (8.14). For the data terms we compute:

$$d_k + 2^{-7k}\int_{\tau}^{1}\sum_{l=x}^{k-1}\frac{2^{5l}}{(\tau')^3}d_l\mathbf{1}_{l,\tau'}d\tau' \lesssim_{\delta} d_k + \sum_{l=x}^{k-1}2^{5(l-k)}d_l \lesssim_{\delta} d_k.$$

In this inequality, we used the fact that $\sum_k\sum_l 2^{-5|k-l|}d_l \lesssim \mathcal{D}_{II}$, so the term $\sum_l 2^{-5|k-l|}d_l$ can be written schematically as d_k. We will use similar bounds implicitly for the rest of the proof.

Additionally, we have the following bound for the discrete error terms S_k :

$$S_k(\tau) + 2^{-7k}\int_{\tau}^{1}\sum_{l=x}^{k-1}\frac{2^{5l}}{(\tau')^3}S_l(\tau')\mathbf{1}_{l,\tau'}d\tau' \lesssim$$

$$\lesssim S_k(\tau) + \delta X 2^{-7k}\int_{\tau}^{1}\sum_{l=x}^{k-1}\sum_{\tau'<X2^{-m-1}\leq 1}\frac{2^{-3l+6m}}{(\tau')^6}a_m(X2^{-m-1})\mathbf{1}_{l,\tau'}d\tau'$$

$$\lesssim S_k(\tau) + \delta X 2^{-7k}\sum_{l=x}^{k-1}\sum_{\substack{\tau<X2^{-m-1}\\ x\leq m\leq l}}\int_{X2^{-l-1}}^{X2^{-m-1}}\frac{2^{-3l+6m}}{(\tau')^6}a_m(X2^{-m-1})d\tau'$$

$$\lesssim S_k(\tau) + \frac{\delta}{X^4}2^{-7k}\sum_{l=x}^{k-1}\sum_{\substack{\tau<X2^{-m-1}\\ x\leq m\leq l}}2^{2l+6m}a_m(X2^{-m-1}) \lesssim \frac{\delta}{X^2}2^{-5k}\sum_{\tau<X2^{-m-1}\leq 1}2^{6m}a_m(X2^{-m-1}).$$

Discrete Grönwall inequality. We deal with the sum of discrete error terms obtained in (8.14) using the discrete Grönwall inequality. We first write the estimate in a convenient form by defining:

$$\widetilde{S}_k(\tau) := \frac{\delta}{X^2}2^{-5k}\sum_{\tau<X2^{-m-1}\leq 1}2^{6m}a_m(X2^{-m-1}),$$

$$\widetilde{E}_k(\tau) := E_k(\tau) + 2^{-7k} \int_\tau^1 \sum_{l=x}^{k-1} \frac{2^{5l}}{(\tau')^3} E_l(\tau') \mathbf{1}_{l,\tau'} d\tau',$$

$$\widetilde{I}_k(\tau) := I_k(\tau) + 2^{-7k} \int_\tau^1 \sum_{l=x}^{k-1} \frac{2^{5l}}{(\tau')^3} I_l(\tau') \mathbf{1}_{l,\tau'} d\tau'.$$

Thus, (8.14) implies that for all $\tau \in [X2^{-k-1}, 1]$:

$$2^k a_k(\tau) \mathbf{1}_{k,\tau} \lesssim C_\delta d_k + \widetilde{S}_k(\tau) + C_\delta \widetilde{E}_k(\tau) + C_\delta \widetilde{I}_k(\tau). \tag{8.15}$$

In particular, there exist constants $C''', C_\delta > 0$ such that for any $k \geq x$:

$$2^k a_k(X2^{-k-1}) \leq C_\delta d_k + C_\delta \widetilde{E}_k(X2^{-k-1}) + C_\delta \widetilde{I}_k(X2^{-k-1}) + C''' \frac{\delta}{X^2} 2^{-5k} \sum_{m=x}^{k-1} 2^{6m} a_m(X2^{-m-1}).$$

We fix the parameters $X, \delta > 0$ satisfying (8.9), (8.10), and :

$$C''' \frac{\delta}{X^2} < \frac{1}{10}. \tag{8.16}$$

We also introduce the notation:

$$b_k = C_\delta d_k + C_\delta \widetilde{E}_k(X2^{-k-1}) + C_\delta \widetilde{I}_k(X2^{-k-1}).$$

Using this notation, we have for any $k \geq x$:

$$2^k a_k(X2^{-k-1}) \leq b_k + \frac{1}{10} 2^{-5k} \sum_{m=x}^{k-1} 2^{5m} \cdot 2^m a_m(X2^{-m-1}).$$

We apply the discrete Grönwall inequality according to [Jon64]:

$$2^k a_k(X2^{-k-1}) \leq b_k + \frac{1}{10} 2^{-5k} \sum_{m=x}^{k-1} \left(2^{5m} b_m \cdot \prod_{j=m+1}^{k-1} \left(1 + 1/10 \cdot 2^{-5j} \cdot 2^{5j}\right) \right) \lesssim b_k + 2^{-4k} \sum_{m=x}^{k-1} 2^{4m} b_m.$$

As a consequence, we have the following bound for all $k \geq x$:

$$\begin{aligned} 2^k a_k(X2^{-k-1}) \lesssim_\delta & \, d_k + \widetilde{E}_k(X2^{-k-1}) + \widetilde{I}_k(X2^{-k-1}) \\ & + \sum_{m=x}^{k-1} 2^{-4k+4m} \Big(\widetilde{E}_m(X2^{-m-1}) + \widetilde{I}_m(X2^{-m-1}) \Big). \end{aligned} \tag{8.17}$$

Notation. We used the parameters $X > 0$ and $\delta > 0$ to apply Grönwall-like inequalities in the above proof. We now fixed these parameters (depending on M, C_0, C_2), so we return to our usual notation convention that we write $A \lesssim B$ if there exists a constant $C > 0$ depending only on the constants M, C_0, C_2 such that $A \leq CB$.

Consequences of the discrete Grönwall inequality. We use the estimate (8.17) in the high frequency regime estimate (8.15). From (8.17) we obtain for all $\tau \in [X2^{-k-1}, 1]$:

$$\widetilde{S}_k(\tau) \lesssim \sum_{\tau < X2^{-m-1} \leq 1} 2^{-5k+5m} d_m + \sum_{\tau < X2^{-m-1} \leq 1} 2^{-4k+4m} \Big(\widetilde{E}_m(X2^{-m-1}) + \widetilde{I}_m(X2^{-m-1}) \Big)$$

$$\lesssim \sum_{\tau < X2^{-m-1} \leq 1} 2^{-5k+5m} d_m + \sum_{\tau < X2^{-m-1} \leq 1} 2^{-4k+4m} \Big(E_m(X2^{-m-1}) + I_m(X2^{-m-1}) \Big),$$

where the second bound follows by:

$$\sum_{\tau < X2^{-m-1} \leq 1} 2^{-4k+4m} \cdot 2^{-7m} \sum_{l=x}^{m-1} \int_{X2^{-l-1}}^{1} \frac{2^{5l}}{(\tau')^3} E_l(\tau') d\tau' \lesssim \sum_{\tau < X2^{-l-1} \leq 1} 2^{-4k+4l} E_l(X2^{-l-1}),$$

and we have a similar inequality for the inhomogeneous terms I_l. We also bound the remaining terms of (8.15):

$$\widetilde{E}_k(\tau) = E_k(\tau) + 2^{-7k} \int_{\tau}^{1} \sum_{l=x}^{k-1} \frac{2^{5l}}{(\tau')^3} E_l(\tau') \mathbf{1}_{l,\tau'} d\tau' \lesssim E_k(\tau) + 2^{-2k} \tau^{-2} \sum_{l \geq x} \int_{\tau}^{1} e_l(\tau') d\tau'$$

$$\lesssim E_k(\tau) + 2^{-2k} \tau^{-2} \int_{\tau}^{1} (\tau')^3 \Big(\big\| \xi \big\|_{H^{3/2}}^2 + \big\| \nabla_\tau \xi \big\|_{H^{1/2}}^2 \Big) d\tau',$$

$$\widetilde{I}_k(\tau) = I_k(\tau) + 2^{-7k} \int_{\tau}^{1} \sum_{l=x}^{k-1} \frac{2^{5l}}{(\tau')^3} I_l(\tau') \mathbf{1}_{l,\tau'} d\tau' \lesssim I_k(\tau) + 2^{-2k} \tau^{-2} \int_{\tau}^{1} \tau' \big\| F_M' \big\|_{H^{1/2}}^2 d\tau'.$$

We combine these estimates and we obtain that (8.15) implies for all $\tau \in [X2^{-k-1}, 1]$:

$$2^k a_k(\tau) \mathbf{1}_{k,\tau} \lesssim d_k + \sum_{\tau < X2^{-m-1} \leq 1} 2^{-5k+5m} d_m + \sum_{\tau < X2^{-m-1} \leq 1} 2^{-4k+4m} \Big(E_m(X2^{-m-1}) + I_m(X2^{-m-1}) \Big)$$

$$+ E_k(\tau) + I_k(\tau) + 2^{-2k} \tau^{-2} \int_{\tau}^{1} (\tau')^3 \Big(\big\| \xi \big\|_{H^{3/2}}^2 + \big\| \nabla_\tau \xi \big\|_{H^{1/2}}^2 \Big) d\tau' + 2^{-2k} \tau^{-2} \int_{\tau}^{1} \tau' \big\| F_M' \big\|_{H^{1/2}}^2 d\tau'.$$

Summing the high frequency estimates. For each $\tau \in (0, 1]$, we sum the above high frequency regime estimates for all $k \geq x$ such that $X2^{-k-1} \leq \tau$:

$$\sum_{\tau \geq X2^{-k-1}} 2^k a_k(\tau) \lesssim \mathcal{D}_{II} + \sum_{\tau < X2^{-m-1} \leq 1} \Big(E_m(X2^{-m-1}) + I_m(X2^{-m-1}) \Big) +$$

$$+ \sum_{\tau \geq X2^{-k-1}} \Big(E_k(\tau) + I_k(\tau) \Big) + \int_{\tau}^{1} (\tau')^3 \Big(\big\| \xi \big\|_{H^{3/2}}^2 + \big\| \nabla_\tau \xi \big\|_{H^{1/2}}^2 \Big) d\tau' + \int_{\tau}^{1} \tau' \big\| F_M' \big\|_{H^{1/2}}^2 d\tau'.$$

Additionally, we have the estimates:

$$\sum_{\tau \geq X2^{-k-1}} E_k(\tau) + \sum_{\tau < X2^{-m-1} \leq 1} E_m(X2^{-m-1}) \lesssim \sum_{k \geq x} \int_{\tau}^{1} e_k(\tau')$$

$$\lesssim \int_\tau^1 (\tau')^3\Big(\big\|\xi\big\|^2_{H^{3/2}} + \big\|\nabla_\tau\xi\big\|^2_{H^{1/2}}\Big)d\tau',$$

$$\sum_{\tau\geq X2^{-k-1}} I_k(\tau) + \sum_{\tau<X2^{-m-1}\leq 1} I_m(X2^{-m-1}) \lesssim \int_\tau^1 \tau'\big\|F'_M\big\|^2_{H^{1/2}}d\tau'.$$

Therefore, we obtain the following high frequency regime estimate:

$$\sum_{\tau\geq X2^{-k-1}} 2^k a_k(\tau) \lesssim \mathcal{D}_{II} + \int_\tau^1 (\tau')^3\Big(\big\|\xi\big\|^2_{H^{3/2}} + \big\|\nabla_\tau\xi\big\|^2_{H^{1/2}}\Big)d\tau' + \int_\tau^1 \tau'\big\|F'_M\big\|^2_{H^{1/2}}d\tau'. \tag{8.18}$$

Proof of Proposition 8.6. The final step in the proof of the optimal high frequency regime estimate (8.8) consists of bounding the second error term on the RHS of (8.18).

We first notice that we have the estimate for all $\tau\in(0,1]$:

$$\tau^3\Big(\big\|\xi\big\|^2_{H^{3/2}} + \big\|\nabla_\tau\xi\big\|^2_{H^{1/2}}\Big) \lesssim \tau^3\Big(\big\|\xi\big\|^2_{H^1} + \big\|\nabla_\tau\xi\big\|^2_{L^2}\Big) + \sum_{k\geq x}\tau^3 2^k\Big(\big\|\nabla P_k\xi\big\|^2_{L^2} + \big\|P_k\nabla_\tau\xi\big\|^2_{L^2}\Big)$$

$$\lesssim \tau^3\Big(\big\|\xi\big\|^2_{H^1} + \big\|\nabla_\tau\xi\big\|^2_{L^2}\Big) + \sum_{\tau<X2^{-k-1}\leq 1}\tau^3 2^k\Big(\big\|\nabla P_k\xi\big\|^2_{L^2} + \big\|P_k\nabla_\tau\xi\big\|^2_{L^2}\Big) + \tau^2\sum_{\tau\geq X2^{-k-1}} 2^k a_k(\tau)$$

$$\lesssim \tau^3\Big(\big\|\xi\big\|^2_{H^1} + \big\|\nabla_\tau\xi\big\|^2_{L^2}\Big) + \sum_{\tau<X2^{-k-1}\leq 1}\tau^2\Big(\big\|\nabla P_k\xi\big\|^2_{L^2} + \big\|P_k\nabla_\tau\xi\big\|^2_{L^2}\Big) + \tau^2\sum_{\tau\geq X2^{-k-1}} 2^k a_k(\tau)$$

$$\lesssim \tau^2\Big(\big\|\xi\big\|^2_{H^1} + \big\|\nabla_\tau\xi\big\|^2_{L^2}\Big) + \tau^2\sum_{\tau\geq X2^{-k-1}} 2^k a_k(\tau).$$

We use the preliminary estimates in Proposition 8.2 and (8.18) to get:

$$\tau^3\Big(\big\|\xi\big\|^2_{H^{3/2}} + \big\|\nabla_\tau\xi\big\|^2_{H^{1/2}}\Big) \lesssim \mathcal{D}_{II} + \int_\tau^1 (\tau')^3\Big(\big\|\xi\big\|^2_{H^{3/2}} + \big\|\nabla_\tau\xi\big\|^2_{H^{1/2}}\Big)d\tau' + \int_\tau^1 \tau'\big\|F'_M\big\|^2_{H^{1/2}}d\tau'.$$

Using Grönwall, we proved that for all $\tau\in(0,1]$:

$$\tau^3\Big(\big\|\xi\big\|^2_{H^{3/2}} + \big\|\nabla_\tau\xi\big\|^2_{H^{1/2}}\Big) \lesssim \mathcal{D}_{II} + \int_\tau^1 \tau'\big\|F'_M\big\|^2_{H^{1/2}}d\tau'. \tag{8.19}$$

Finally, we use this estimate in (8.18) to obtain the optimal high frequency estimate (8.8).

8.6 The Main Estimate in Theorem 8.1

In this section, we prove the main estimate (8.2) in Theorem 8.1. For this, we first establish the following top order estimate for the singular component Φ_0 :

$$\mathcal{E}^0_{II}(\tau) \lesssim \mathcal{D}_{II} + \sum_{m=0}^M \int_\tau^1 \tau'\big\|F'_m\big\|^2_{H^{1/2}}d\tau', \tag{8.20}$$

where we define the top order energy for the singular component:

$$\mathcal{E}^0_{II}(\tau) := \tau\big\|\Phi_0\big\|^2_{H^{M+1/2}} + \tau^2\big\|\Phi_0\big\|^2_{H^{M+3/2}} + \tau^2\big\|\nabla_\tau\nabla^M\Phi_0\big\|^2_{H^{1/2}} + \\ + \sum_{m=0}^{M-1}\tau^2\big\|\nabla_\tau\nabla^m\Phi_0\big\|^2_{L^2} + \int_\tau^1 \tau'\big\|\Phi_0\big\|^2_{H^{M+1}}d\tau'.$$

Using Propositions 8.2, 8.4 and the estimate (8.8), we have the bound for all $\tau \in (0,1]$:

$$\tau^2\Big(\big\|\xi\big\|^2_{H^{3/2}} + \big\|\nabla_\tau\xi\big\|^2_{H^{1/2}}\Big) \lesssim \tau^2\Big(\big\|\xi\big\|^2_{H^1} + \big\|\nabla_\tau\xi\big\|^2_{L^2}\Big) + \sum_{k\geq x}\tau^2 2^k\Big(\big\|\nabla P_k\xi\big\|^2_{L^2} + \big\|P_k\nabla_\tau\xi\big\|^2_{L^2}\Big)$$

$$\lesssim \tau^2\Big(\big\|\xi\big\|^2_{H^1} + \big\|\nabla_\tau\xi\big\|^2_{L^2}\Big) + \sum_{\tau<X2^{-k-1}\leq 1}\tau^2 2^k\Big(\big\|\nabla P_k\xi\big\|^2_{L^2} + \big\|P_k\nabla_\tau\xi\big\|^2_{L^2}\Big) + \tau\sum_{\tau\geq X2^{-k-1}} 2^k a_k(\tau)$$

$$\lesssim \mathcal{D}_{II} + \int_\tau^1 \tau'\big\|F'_M\big\|^2_{H^{1/2}}d\tau' + \sum_{\tau<X2^{-k-1}\leq 1} 2^k a_k(X2^{-k-1}).$$

We use the bound (8.17) for $2^k a_k(X2^{-k-1})$ to get:

$$\sum_{\tau<X2^{-k-1}\leq 1} 2^k a_k(X2^{-k-1}) \lesssim \mathcal{D}_{II} + \sum_{\tau<X2^{-k-1}\leq 1}\widetilde{E}_k(X2^{-k-1}) + \sum_{\tau<X2^{-k-1}\leq 1}\widetilde{I}_k(X2^{-k-1})$$

$$\lesssim \mathcal{D}_{II} + \sum_{\tau<X2^{-k-1}\leq 1} E_k(X2^{-k-1}) + \sum_{\tau<X2^{-k-1}\leq 1} I_k(X2^{-k-1}) \lesssim \mathcal{D}_{II} + \int_\tau^1 \tau'\big\|F'_M\big\|^2_{H^{1/2}}d\tau'.$$

We note that we used (8.19) in the last inequality. As a result, we proved that:

$$\tau^2\Big(\big\|\xi\big\|^2_{H^{3/2}} + \big\|\nabla_\tau\xi\big\|^2_{H^{1/2}}\Big) \lesssim \mathcal{D}_{II} + \int_\tau^1 \tau'\big\|F'_M\big\|^2_{H^{1/2}}d\tau'.$$

As usual, we also get for all $\tau \in (0,1]$:

$$\tau\big\|\xi\big\|^2_{H^{1/2}} \lesssim \mathcal{D}_{II} + \int_\tau^1 \tau'\big\|\xi\big\|_{H^{1/2}}\cdot\big\|\nabla_\tau\xi\big\|_{H^{1/2}}d\tau'$$

$$\lesssim \mathcal{D}_{II} + \int_\tau^1 \sqrt{\tau'}\big\|\xi\big\|^2_{H^{1/2}}d\tau' + \int_\tau^1 (\tau')^{3/2}\big\|\nabla_\tau\xi\big\|^2_{H^{1/2}}d\tau'$$

$$\lesssim \mathcal{D}_{II} + \int_\tau^1 \tau'\big\|F'_M\big\|^2_{H^{1/2}}d\tau' + \int_\tau^1 \sqrt{\tau'}\big\|\xi\big\|^2_{H^{1/2}}d\tau'.$$

Applying Grönwall, we proved that:

$$\tau^2\big\|\xi\big\|^2_{H^{3/2}} + \tau^2\big\|\nabla_\tau\xi\big\|^2_{H^{1/2}} + \tau\big\|\xi\big\|^2_{H^{1/2}} \lesssim \mathcal{D}_{II} + \int_\tau^1 \tau'\big\|F'_M\big\|^2_{H^{1/2}}d\tau'.$$

We recall that $\xi = \nabla^M\Phi_0$. Combining the above estimate with the preliminary estimates in Proposition 8.2, we proved (8.20).

Proof of (8.2). We use the estimates for the regular quantities in Section 8.1 and the top order estimate for the singular quantities (8.20) to get:

$$\mathcal{E}_{II}(\tau) \lesssim \mathcal{D}_{II} + \mathcal{F}_{II}(\tau) + \sum_{m=0}^{M} \int_{\tau}^{1} \tau' \big\| F'_m \big\|^2_{H^{1/2}} d\tau'.$$

We also recall our notation $F'_m = \psi \sum_{i=1}^{I} \nabla^{m+1}\Phi_i + F^0_m$. Using again the estimates for the regular quantities proved in Section 8.1, we conclude. ☐

8.7 Estimates for the Asymptotic Quantities

In this section we prove the estimates (8.3) and (8.4) for the asymptotic quantities $\mathcal{O}$, $\mathfrak{h}$, and Φ^0_i with $1 \leq i \leq I$, in order to complete the proof of Theorem 8.1. We notice that the estimates for the regular quantities in Proposition 8.1 already imply the bound (1.45) for Φ^0_i with $1 \leq i \leq I$. As a result, we only need to prove the estimates (1.48) and (8.4) for $\mathcal{O}$ and $\mathfrak{h}$.

Proposition 8.7. *The obstruction tensor* $\mathcal{O}$ *satisfies the estimate* (1.48)*:*

$$\big\|\mathcal{O}\big\|^2_{H^{M+1}} \lesssim \mathcal{D}_{II} + \mathcal{F}_{II}(0).$$

Proof. As a consequence of the preliminary estimates in Proposition 8.2, we obtain for all $0 \leq m \leq M$:

$$\big\|\nabla^m \mathcal{O}\big\|^2_{L^2} \lesssim \Big(\big\|\Phi_0\big\|^2_{H^{m+3/2}} + \big\|\nabla_\tau \Phi_0\big\|^2_{H^{m+1/2}} \Big)\Big|_{\tau=1} + \int_0^1 (\tau')^2 \big\|F'_m\big\|^2_{L^2} d\tau' \lesssim \mathcal{D}_{II} + \mathcal{F}_{II}(0),$$

so it remains to prove the above bound for $\big\|\nabla^{M+1}\mathcal{O}\big\|^2_{L^2}$.

For any $k \geq x$ and $0 \leq \tau < X 2^{-k-1} \leq 1$, the low frequency regime estimate in Proposition 8.4 implies:

$$2^{2k} \big\|\tau P_k \nabla_\tau \nabla^M \Phi_0\big\|^2_{L^2} \lesssim 2^k a_k (X 2^{-k-1}) + 2^{-k}\mathcal{D}_{II} +$$
$$+ 2^k \int_\tau^1 (\tau')^2 \big\|P_k F'_M\big\|^2_{L^2} d\tau' + 2^{-k} \int_\tau^1 (\tau')^2 \big\|F'_M\big\|^2_{L^2} d\tau'.$$

Using the expansion of $\nabla_\tau \nabla^M \Phi_0$ at $\mathcal{I}^-$, we get that for all $k \geq x$:

$$2^{2k} \big\|P_k \nabla^M \mathcal{O}\big\|^2_{L^2} \lesssim 2^k a_k (X 2^{-k-1}) + 2^{-k}\mathcal{D}_{II} + 2^k \int_0^1 (\tau')^2 \big\|P_k F'_M\big\|^2_{L^2} d\tau' + 2^{-k} \int_0^1 (\tau')^2 \big\|F'_M\big\|^2_{L^2}.$$

Together with our previous estimate for $\|\mathcal{O}\|_{H^M}$, we obtain that:

$$\big\|\nabla^M \mathcal{O}\big\|^2_{H^1} \lesssim \mathcal{D}_{II} + \mathcal{F}_{II}(0) + \sum_{k \geq x} 2^k a_k (X 2^{-k-1}).$$

We complete the proof since we already proved in Section 8.6:

$$\sum_{k\geq x}2^k a_k(X2^{-k-1})\lesssim\mathcal{D}_{II}+\int_0^1\tau'\big\|F'_M\big\|^2_{H^{1/2}}d\tau'\lesssim\mathcal{D}_{II}+\mathcal{F}_{II}(0).$$

□

In order to prove the estimate (8.4) for $\mathfrak{h}$, we first prove that it can be reduced to the proof of (1.49). We recall the notation $\mathfrak{h}_M=\nabla^M h-2(\log\nabla)\nabla^M\mathcal{O}$, and notice that we have the bounds:

$$\begin{aligned}\big\|\mathfrak{h}\big\|^2_{H^{M+1}}&\lesssim\big\|\mathfrak{h}\big\|^2_{L^2}+\big\|\big[(\log\nabla),\nabla^M\big]\mathcal{O}\big\|^2_{H^1}+\big\|\nabla\mathfrak{h}_M\big\|^2_{L^2}\\&\lesssim\big\|\mathfrak{h}\big\|^2_{L^2}+C\Big(\big\|Ri\not{e}m_0\big\|^2_{H^M}\Big)\big\|\mathcal{O}\big\|^2_{H^M}+\sum_{k\geq x}2^{2k}\big\|P_k\mathfrak{h}_M\big\|^2_{L^2}+\big\|\mathfrak{h}_M\big\|^2_{L^2}\\&\lesssim\big\|\mathfrak{h}\big\|^2_{H^M}+C\Big(\big\|Ri\not{e}m_0\big\|^2_{H^M}\Big)\big\|\mathcal{O}\big\|^2_{H^M}+\sum_{k\geq x}2^{2k}\big\|P_k\mathfrak{h}_M\big\|^2_{L^2},\end{aligned}$$

where we used the proof of Lemma 7.2 to bound the commutator term. We use the interpolation inequalities of [KR06, Proposition 7.7] and the weighted AM-GM inequality for the first term to obtain:

$$\big\|\mathfrak{h}\big\|^2_{H^{M+1}}\lesssim\big\|\mathfrak{h}\big\|^2_{L^2}+C\Big(\big\|Ri\not{e}m_0\big\|^2_{H^M}\Big)\big\|\mathcal{O}\big\|^2_{H^M}+\sum_{k\geq x}2^{2k}\big\|P_k\mathfrak{h}_M\big\|^2_{L^2}.$$

To deal with the first term, we use the bound in Lemma 6.7:

$$\big\|\mathfrak{h}\big\|^2_{H^{M+1}}\lesssim\big\|h\big\|^2_{L^2}+C\Big(\big\|Ri\not{e}m_0\big\|^2_{H^M}\Big)\big\|\mathcal{O}\big\|^2_{H^M}+\sum_{k\geq x}2^{2k}\big\|P_k\mathfrak{h}_M\big\|^2_{L^2}.$$

Using the already established bound for $\mathcal{O}$, we showed that it suffices to prove (1.49):

$$\sum_{k\geq x}2^{2k}\big\|P_k\mathfrak{h}_M\big\|^2_{L^2}\lesssim\mathcal{D}_{II}+\mathcal{F}_{II}(0).$$

We recall our notation $\xi=\nabla^M\Phi_0$ and we consider the renormalized quantities for all $k\geq x$:

$$\overline{\xi}_k=P_k\xi-\tau\log(2^k\tau)P_k\nabla_\tau\xi.$$

Using the expansion for ξ at $\mathcal{I}^-$, we get:

$$\lim_{\tau\to0}\overline{\xi}_k=\lim_{\tau\to0}\Big(P_k\nabla^M\Phi_0-\tau\log(2^k\tau)P_k\nabla_\tau\nabla^M\Phi_0\Big)=P_k\nabla^M h-2\log(2^k)P_k\nabla^M\mathcal{O}.$$

Recalling the definition of the operator R_k in (6.30), we can write:

$$P_k\nabla^M h-2\log(2^k)P_k\nabla^M\mathcal{O}=P_k\mathfrak{h}_M+2P_k(\log\nabla)\nabla^M\mathcal{O}-2\log(2^k)P_k\nabla^M\mathcal{O}=P_k\mathfrak{h}_M+R_k\nabla^M\mathcal{O}.$$

Thus, we obtain using (6.33):

$$2^{2k}\big\|P_k\mathfrak{h}_M\big\|_{L^2}^2 \lesssim 2^{2k}\lim_{\tau\to 0}\big\|\overline{\xi}_k\big\|_{L^2}^2 + 2^{2k}\big\|R_k\nabla^M\mathcal{O}\big\|_{L^2}^2 \lesssim 2^{2k}\lim_{\tau\to 0}\big\|\overline{\xi}_k\big\|_{L^2}^2 + \big\|\underline{P}_k\nabla^M\mathcal{O}\big\|_{H^1}^2.$$

Summing for all $k\geq x$, we get that:

$$\sum_{k\geq x}2^{2k}\big\|P_k\mathfrak{h}_M\big\|_{L^2}^2 \lesssim \big\|\mathcal{O}\big\|_{H^{M+1}}^2 + \sum_{k\geq x}2^{2k}\lim_{\tau\to 0}\big\|\overline{\xi}_k\big\|_{L^2}^2.$$

According to the above estimate and (1.48), in order to prove (1.49) we need to establish the following result:

Proposition 8.8. *We have the following estimate:*

$$\sum_{k\geq x}2^{2k}\lim_{\tau\to 0}\big\|\overline{\xi}_k\big\|_{L^2}^2 \lesssim \mathcal{D}_{II} + \mathcal{F}_{II}(0).$$

As a result, (8.4) *holds, completing the proof of Theorem 8.1.*

Proof. Using (8.7) as in the proof of Proposition 8.4, we get that $\overline{\xi}_k$ satisfies the equation on $\tau\in(0,X2^{-k-1}]$:

$$\nabla_\tau\overline{\xi}_k = [\nabla_\tau,P_k]\xi - \log(2^k\tau)\nabla_\tau\big(\tau P_k\nabla_\tau\xi\big) =$$

$$= \tau[\nabla_4,P_k]\xi - \log(2^k\tau)\Big(4\tau\Delta P_k\xi + \tau P_k\big(\psi\nabla\xi\big) + \tau[\nabla_4,P_k]\tau\nabla_\tau\xi + \tau P_kF_M'\Big).$$

We contract by $\overline{\xi}_k$ to obtain the energy estimate:

$$2^{2k}\big\|\overline{\xi}_k\big\|_{L^2}^2 \lesssim \Big(2^{2k}\big\|P_k\xi\big\|_{L^2}^2 + \big\|P_k\nabla_\tau\xi\big\|_{L^2}^2\Big)\Big|_{\tau=X2^{-k-1}} + \int_\tau^{X2^{-k-1}}2^k\big\|\overline{\xi}_k\big\|_{L^2}\cdot 2^k\tau'\big\|[\nabla_4,P_k]\xi\big\|_{L^2} +$$

$$+\int_\tau^{X2^{-k-1}}2^k\big\|\overline{\xi}_k\big\|_{L^2}\cdot|2^k\tau'\log(2^k\tau')|\cdot\big\|\Delta P_k\xi\big\|_{L^2} +$$

$$+\int_\tau^{X2^{-k-1}}2^k\big\|\overline{\xi}_k\big\|_{L^2}\cdot|2^k\tau'\log(2^k\tau')|\cdot\big\|P_k\big(\psi\nabla\xi\big)\big\|_{L^2} +$$

$$+\int_\tau^{X2^{-k-1}}2^k\big\|\overline{\xi}_k\big\|_{L^2}\cdot|2^k\tau'\log(2^k\tau')|\cdot\big\|[\nabla_4,P_k]\tau'\nabla_\tau\xi\big\|_{L^2} +$$

$$+\int_\tau^{X2^{-k-1}}2^k\big\|\overline{\xi}_k\big\|_{L^2}\cdot|2^k\tau'\log(2^k\tau')|\cdot\big\|P_kF_M'\big\|_{L^2}.$$

We use Lemma 6.3, Grönwall, and the notation in Section 8.5 to obtain:

$$2^{2k}\big\|\overline{\xi}_k\big\|_{L^2}^2 \lesssim 2^ka_k(X2^{-k-1}) + \int_\tau^{X2^{-k-1}}2^k(\tau')^2\big\|\xi\big\|_{L^2}^2 + \int_\tau^{X2^{-k-1}}2^k(\tau')^2|\log(2^k\tau')|^2\big\|\Delta P_k\xi\big\|_{L^2}^2$$

$$+\int_\tau^{X2^{-k-1}}2^{-k}(\tau')^2|\log(2^k\tau')|^2\big\|\xi\big\|_{H^1}^2 + \int_\tau^{X2^{-k-1}}2^k(\tau')^2|\log(2^k\tau')|^2\big\|\nabla P_k\xi\big\|_{L^2}^2$$

$$+\int_{\tau}^{X2^{-k-1}} 2^k(\tau')^2|\log(2^k\tau')|^2\big\|\tau'\nabla_\tau\xi\big\|_{L^2}^2+\int_{\tau}^{X2^{-k-1}} 2^k(\tau')^2|\log(2^k\tau')|^2\big\|P_kF_M'\big\|_{L^2}^2.$$

We use the preliminary estimates in Proposition 8.2 to get:

$$2^{2k}\big\|\overline{\xi}_k\big\|_{L^2}^2\lesssim 2^{-k}\mathcal{D}_{II}+2^ka_k(X2^{-k-1})+$$

$$+\int_{\tau}^{X2^{-k-1}} 2^k(\tau')^2|\log(2^k\tau')|^2\big\|\Delta P_k\xi\big\|_{L^2}^2d\tau'+2^{-k}\int_{\tau}^{1}(\tau')^2\big\|F_M'\big\|_{L^2}^2d\tau'+\int_{\tau}^{1}\tau'\big\|P_kF_M'\big\|_{L^2}^2d\tau'.$$

Using the finite band property of the LP projections for $P_k=\underline{P}_k^2$, we have the bound:

$$\int_{\tau}^{X2^{-k-1}} 2^k(\tau')^2|\log(2^k\tau')|^2\big\|\Delta P_k\xi\big\|_{L^2}^2d\tau'\lesssim\int_{\tau}^{X2^{-k-1}} 2^{3k}(2^k\tau')^2|\log(2^k\tau')|^2\big\|\underline{P}_k\xi\big\|_{L^2}^2d\tau'. \tag{8.21}$$

Once we established the above estimate for $2^k\|\overline{\xi}_k\|_{L^2}$, the idea of the proof is to decompose the error term in (8.21) into its low frequency and high frequency regime parts, in order to use our previous estimates. For each $\tau'\in[\tau,X2^{-k-1}]$ we have:

$$\big\|\underline{P}_k\xi\big\|_{L^2}(\tau')\lesssim\sum_{l<x}\big\|P_l^2\underline{P}_k\xi\big\|_{L^2}+\sum_{l=x}^{k-1}\big\|P_l^2\underline{P}_k\xi\big\|_{L^2}+\sum_{\substack{l\geq k\\ \tau'<X2^{-l-1}}}\big\|P_l^2\underline{P}_k\xi\big\|_{L^2}+\sum_{l\geq k}\mathbf{1}_{l,\tau'}\big\|P_l^2\underline{P}_k\xi\big\|_{L^2}. \tag{8.22}$$

The first three terms in (8.22) are in the low frequency regime, and the last term is in the high frequency regime.

We bound the *first term* in (8.22) using the L^2 almost orthogonality of the LP projections:

$$\sum_{l<x}\big\|P_l^2\underline{P}_k\xi\big\|_{L^2}(\tau')\lesssim\sum_{l<x}2^{-5k+5l}\big\|P_l\xi\big\|_{L^2}(\tau')\lesssim 2^{-5k}\big\|\xi\big\|_{L^2}(\tau').$$

The corresponding term in (8.21) is bounded using Proposition 8.2:

$$\int_{\tau}^{X2^{-k-1}} 2^{-6k}(2^k\tau')|\log(2^k\tau')|^2\cdot\tau'\big\|\xi\big\|_{L^2}^2d\tau'\lesssim 2^{-k}\mathcal{D}_{II}+2^{-k}\mathcal{F}_{II}(\tau).$$

For the *second term* in (8.22) we use Cauchy-Schwarz and we consider a projection operator with $\underline{P}_k=\widetilde{\underline{P}}_k^2$:

$$\Big(\sum_{l=x}^{k-1}\big\|P_l^2\widetilde{\underline{P}}_k^2\xi\big\|_{L^2}(\tau')\Big)^2\lesssim\sum_{l=x}^{k-1}2^{-5(k-l)}\big\|\widetilde{\underline{P}}_kP_l\xi\big\|_{L^2}^2(\tau')\lesssim\sum_{l=x}^{k-1}2^{-2k}2^{-5(k-l)}\big\|\nabla P_l\xi\big\|_{L^2}^2(\tau')$$

$$\lesssim\frac{1}{(\tau')^2}\sum_{l=x}^{k-1}2^{-2k}2^{-5(k-l)}\big(2^{-2l}\cdot 2^la_l(X2^{-l-1})+2^{-3l}\mathcal{D}_{II}+2^{-3l}\mathcal{F}_{II}(\tau')\big),$$

where we used Proposition 8.4 in the second line. The corresponding term in (8.21) is bounded by:

$$\int_{\tau}^{X2^{-k-1}} \sum_{l=x}^{k-1} 2^{k}|\log(2^{k}\tau')|^{2} 2^{-3(k-l)}\Big(2^{l}a_{l}(X2^{-l-1}) + 2^{-l}\mathcal{D}_{II} + 2^{-l}\mathcal{F}_{II}(\tau')\Big)d\tau' \lesssim$$

$$\lesssim 2^{-k}\mathcal{D}_{II} + 2^{-k}\mathcal{F}_{II}(\tau) + \sum_{l=x}^{k-1} 2^{-3(k-l)} \cdot 2^{l}a_{l}(X2^{-l-1}).$$

For the *third term* in (8.22), we have similarly from Proposition 8.4:

$$\Bigg(\sum_{\substack{l\geq k \\ \tau'<X2^{-l-1}}} \big\|P_{l}^{2}\underline{\widetilde{P}}_{k}^{2}\xi\big\|_{L^{2}}(\tau')\Bigg)^{2} \lesssim \sum_{\substack{l\geq k \\ \tau'<X2^{-l-1}}} 2^{-5(l-k)}\big\|\underline{\widetilde{P}}_{k}P_{l}\xi\big\|_{L^{2}}^{2}(\tau')$$

$$\lesssim \sum_{\substack{l\geq k \\ \tau'<X2^{-l-1}}} 2^{-2k}2^{-5(l-k)}\big\|\nabla P_{l}\xi\big\|_{L^{2}}^{2}(\tau')$$

$$\lesssim \frac{1}{(\tau')^{2}} \sum_{\substack{l\geq k \\ \tau'<X2^{-l-1}}} 2^{-2k}2^{-5(l-k)}\big(2^{-2l}\cdot 2^{l}a_{l}(X2^{-l-1}) + 2^{-3l}\mathcal{D}_{II} + 2^{-3l}\mathcal{F}_{II}(\tau')\big).$$

As a result, the corresponding term in (8.21) is bounded using Proposition 8.4:

$$\sum_{\substack{l\geq k \\ \tau<X2^{-l-1}}} \int_{\tau}^{X2^{-l-1}} 2^{k}|\log(2^{k}\tau')|^{2}2^{-7(l-k)}\big(2^{l}a_{l}(X2^{-l-1}) + 2^{-l}\mathcal{D}_{II} + 2^{-l}\mathcal{F}_{II}(\tau')\big)d\tau' \lesssim$$

$$\lesssim \sum_{\substack{l\geq k \\ \tau<X2^{-l-1}}} \big(2^{l}a_{l}(X2^{-l-1}) + 2^{-l}\mathcal{D}_{II} + 2^{-l}\mathcal{F}_{II}(\tau)\big) \cdot \int_{\tau}^{X2^{-l-1}} 2^{k}|\log(2^{k}\tau')|^{2}2^{-7(l-k)}d\tau' \lesssim$$

$$\lesssim 2^{-k}\mathcal{D}_{II} + 2^{-k}\mathcal{F}_{II}(\tau) + \sum_{l\geq k} 2^{-7(l-k)} \cdot 2^{l}a_{l}(X2^{-l-1}).$$

For the high frequency regime *fourth term* in (8.22), we have using our notation in Section 8.5:

$$\Bigg(\sum_{l\geq k}\mathbf{1}_{l,\tau'}\big\|P_{l}^{2}\underline{P}_{k}\xi\big\|_{L^{2}}(\tau')\Bigg)^{2} \lesssim \sum_{l\geq k}\mathbf{1}_{l,\tau'}2^{-5(l-k)}\big\|P_{l}\xi\big\|_{L^{2}}^{2}(\tau') \lesssim \tau'\sum_{l\geq k}\mathbf{1}_{l,\tau'}2^{-5(l-k)}a_{l}(\tau').$$

The corresponding term in (8.21) is bounded in Lemma 8.2 at the end of the section.

We combine the previous bounds and we get from (8.21) that for all $\tau \in (0, X2^{-k-1}]$:

$$2^{2k}\big\|\overline{\xi}_{k}\big\|_{L^{2}}^{2} \lesssim d_{k} + \tau^{6}2^{6k}\mathcal{D}_{II} + 2^{k}a_{k}(X2^{-k-1}) + \sum_{l\geq x} 2^{-3|k-l|}\cdot 2^{l}a_{l}(X2^{-l-1})$$

$$+\big(2^{-k} + \tau^{6}2^{6k}\big)\mathcal{F}_{II}(\tau) + \sum_{l\geq k} 2^{-6(l-k)}\int_{\tau}^{1} 2^{l}\tau'\big\|P_{l}F_{M}'\big\|_{L^{2}}^{2}d\tau' + \widetilde{E}_{k}(X2^{-k-1}) + \widetilde{I}_{k}(X2^{-k-1}) +$$

$$+\sum_{\tau<X2^{-m-1}\le 1} 2^{-4|k-m|}\Big(E_m(X2^{-m-1})+I_m(X2^{-m-1})\Big)+\sum_{l\ge k}2^{-6(l-k)}\int_\tau^1 e_l(\tau')d\tau'.$$

We take the limit $\tau\to 0$ and sum for all $k\ge x$:

$$\sum_{k\ge x}\Big(2^{2k}\lim_{\tau\to 0}\big\|\bar{\xi}_k\big\|_{L^2}^2\Big)\lesssim \mathcal{D}_{II}+\mathcal{F}_{II}(0)+\sum_{k\ge x}2^k a_k(X2^{-k-1})$$

$$+\sum_{k\ge x}\big(\widetilde{E}_k(X2^{-k-1})+\widetilde{I}_k(X2^{-k-1})\big)+\int_0^1(\tau')^3\Big(\big\|\xi\big\|_{H^{3/2}}^2+\big\|\nabla_\tau\xi\big\|_{H^{1/2}}^2\Big)d\tau'.$$

For the last three terms, we use the estimates in Section 8.5 and we obtain the conclusion. □

We conclude the section by proving the additional estimate used in the previous proof:

Lemma 8.2. *We have the bound for all* $\tau\in(0,X2^{-k-1}]$*:*

$$\int_\tau^{X2^{-k-1}}2^k(2^k\tau')^3|\log(2^k\tau')|^2\sum_{l\ge k}\mathbf{1}_{l,\tau'}2^{-6(l-k)}\cdot 2^l a_l(\tau')d\tau'\lesssim$$

$$\lesssim d_k+\tau^6 2^{6k}\cdot\big(\mathcal{D}_{II}+\mathcal{F}_{II}(\tau)\big)+\sum_{\tau<X2^{-m-1}\le 1}2^{-4|k-m|}\Big(E_m(X2^{-m-1})+I_m(X2^{-m-1})\Big)+$$

$$+\widetilde{E}_k(X2^{-k-1})+\widetilde{I}_k(X2^{-k-1})+\sum_{l\ge k}2^{-6(l-k)}\bigg(\int_\tau^1 e_l(\tau')d\tau'+\int_\tau^1 2^l\tau'\big\|P_lF_M'\big\|_{L^2}^2d\tau'\bigg)+2^{-k}\mathcal{F}_{II}(\tau).$$

Proof. We use the estimate (8.15) as a starting point. We also use the bound on $\widetilde{S}_l$ in Section 8.5, and the definitions of $\widetilde{E}_l$ and $\widetilde{I}_l$. Thus, we have for all $\tau\in[X2^{-l-1},1]$:

$$2^l a_l(\tau)\mathbf{1}_{l,\tau}\lesssim d_l+\widetilde{S}_l(\tau)+\widetilde{E}_l(\tau)+\widetilde{I}_l(\tau)$$

$$\lesssim d_l+\sum_{\tau<X2^{-m-1}\le 1}2^{-5l+5m}d_m+\sum_{\tau<X2^{-m-1}\le 1}2^{-4l+4m}\Big(E_m(X2^{-m-1})+I_m(X2^{-m-1})\Big)+$$

$$+E_l(\tau)+I_l(\tau)+2^{-7l}\int_\tau^1\sum_{m=x}^{l-1}\frac{2^{5m}}{(\tau')^3}\Big(E_m(\tau')+I_m(\tau')\Big)\mathbf{1}_{m,\tau'}d\tau'.$$

We use this bound to obtain a total of 8 error terms that control our main integral. We notice that the first two terms can be dealt with in a straightforward way. For the third and fourth terms we compute that:

$$\int_\tau^{X2^{-k-1}}2^k\sum_{l\ge k}\sum_{\tau'<X2^{-m-1}\le 1}\mathbf{1}_{l,\tau'}2^{-6(l-k)}2^{-4l+4m}\Big(E_m(X2^{-m-1})+I_m(X2^{-m-1})\Big)d\tau'\lesssim$$

$$\lesssim\sum_{\tau<X2^{-m-1}\le 1}2^{-4|k-m|}\Big(E_m(X2^{-m-1})+I_m(X2^{-m-1})\Big),$$

where we used the fact that $l \geq m$ and $l \geq k$ in the first line. For the fifth and sixth terms we have:

$$\int_{\tau}^{X2^{-k-1}} 2^k \sum_{l \geq k} \mathbf{1}_{l,\tau'} 2^{-6(l-k)} \Big(E_l(\tau') + I_l(\tau') \Big) d\tau' \lesssim$$

$$\lesssim \sum_{l \geq k} 2^{-6(l-k)} \bigg(\int_{\tau}^{1} e_l(\tau') d\tau' + \int_{\tau}^{1} 2^l \tau' \big\| P_l F'_M \big\|^2_{L^2} d\tau' \bigg) + 2^{-k} \int_{\tau}^{1} \tau' \big\| F'_M \big\|^2_{H^{1/2}} d\tau'.$$

For the seventh and eight terms we can write:

$$\int_{\tau}^{X2^{-k-1}} 2^k \sum_{l \geq k} \mathbf{1}_{l,\tau'} 2^{-6(l-k)} \cdot \bigg(2^{-7l} \int_{\tau'}^{1} \sum_{m=x}^{l-1} \frac{2^{5m}}{(\tau'')^3} \Big(E_m(\tau'') + I_m(\tau'') \Big) \mathbf{1}_{m,\tau''} \bigg) \lesssim I + II + III,$$

where we introduce the notation:

$$I = \int_{\tau}^{X2^{-k-1}} 2^k \sum_{l \geq k} \mathbf{1}_{l,\tau'} 2^{-6(l-k)} \cdot 2^{-7l} \sum_{m=x}^{k-1} \bigg(\int_{\tau'}^{1} \frac{2^{5m}}{(\tau'')^3} \Big(E_m(\tau'') + I_m(\tau'') \Big) \mathbf{1}_{m,\tau''} d\tau'' \bigg) d\tau',$$

$$II = \int_{\tau}^{X2^{-k-1}} 2^k \sum_{l \geq k} \mathbf{1}_{l,\tau'} 2^{-6(l-k)} \cdot 2^{-7l} \sum_{\substack{\tau < X2^{-m-1} \leq 1 \\ k \leq m < l}} \bigg(\int_{\tau'}^{1} \frac{2^{5m}}{(\tau'')^3} \Big(E_m(\tau'') + I_m(\tau'') \Big) \mathbf{1}_{m,\tau''} \bigg),$$

$$III = \int_{\tau}^{X2^{-k-1}} 2^k \sum_{l \geq k} \mathbf{1}_{l,\tau'} 2^{-6(l-k)} \cdot 2^{-7l} \sum_{\substack{\tau > X2^{-m-1} \\ m < l}} \bigg(\int_{\tau'}^{1} \frac{2^{5m}}{(\tau'')^3} \Big(E_m(\tau'') + I_m(\tau'') \Big) d\tau'' \bigg) d\tau'.$$

We first notice that in the inner integral of I we have $\tau'' \geq X2^{-m-1} > X2^{-k-1}$. Thus, we have the bound:

$$I \lesssim 2^{-7k} \sum_{m=x}^{k-1} \bigg(\int_{X2^{-k-1}}^{1} \frac{2^{5m}}{(\tau'')^3} \Big(E_m(\tau'') + I_m(\tau'') \Big) \mathbf{1}_{m,\tau''} d\tau'' \bigg) \lesssim \widetilde{E}_k(X2^{-k-1}) + \widetilde{I}_k(X2^{-k-1}),$$

where in the second inequality we used the definitions of $\widetilde{E}_k$ and $\widetilde{I}_k$. Next, we have the bound:

$$II \lesssim \int_{\tau}^{X2^{-k-1}} 2^k \sum_{l \geq k} \mathbf{1}_{l,\tau'} 2^{6k-13l} \sum_{\substack{\tau < X2^{-m-1} \leq 1 \\ k \leq m < l}} \bigg[\Big[E_m(X2^{-m-1}) + I_m(X2^{-m-1}) \Big] \int_{\tau'}^{1} \frac{2^{5m}}{(\tau'')^3} \mathbf{1}_{m,\tau''} \bigg]$$

$$\lesssim \int_{\tau}^{X2^{-k-1}} 2^k \sum_{l \geq k} \mathbf{1}_{l,\tau'} 2^{6k-13l} \sum_{\substack{\tau < X2^{-m-1} \leq 1 \\ k \leq m < l}} 2^{7m} \Big(E_m(X2^{-m-1}) + I_m(X2^{-m-1}) \Big) d\tau'$$

$$\lesssim \sum_{\tau < X2^{-m-1} \leq X2^{-k-1}} 2^{7m} \Big(E_m(X2^{-m-1}) + I_m(X2^{-m-1}) \Big) \cdot \sum_{l > m} 2^{6k-13l}$$

$$\lesssim \sum_{\tau < X2^{-m-1} \leq X2^{-k-1}} 2^{-6|m-k|} \Big(E_m(X2^{-m-1}) + I_m(X2^{-m-1}) \Big),$$

where we used again $\tau'' \geq X2^{-m-1}$ to compute the inner integral in the first line.

Finally, we notice that the last error term III must vanish as $\tau \to 0$, since the inner sum would be empty. We prove a bound consistent with this expectation:

$$\begin{aligned}
III &\lesssim \sum_{\tau > X2^{-m-1}} \left(\int_\tau^1 \frac{2^{5m}}{(\tau'')^3} \Big(E_m(\tau'') + I_m(\tau'') \Big) d\tau'' \right) \cdot \int_\tau^{X2^{-k-1}} 2^k \sum_{m<l} 2^{-6(l-k)} \cdot 2^{-7l} d\tau' \\
&\lesssim 2^{6k} \sum_{\tau > X2^{-m-1}} 2^{-8m} \left(\int_\tau^1 \frac{1}{(\tau')^3} \Big(E_m(\tau') + I_m(\tau') \Big) d\tau' \right) \\
&\lesssim \int_\tau^1 \Big(\tau' \big\| F_M' \big\|_{H^{1/2}}^2 + (\tau')^3 \big\| \xi \big\|_{H^{3/2}}^2 + (\tau')^3 \big\| \nabla_\tau \xi \big\|_{H^{1/2}}^2 \Big) d\tau' \cdot \tau^{-2} 2^{6k} \sum_{\tau > X2^{-m-1}} 2^{-8m} \\
&\lesssim \tau^6 2^{6k} \left(\mathcal{D}_{II} + \int_\tau^1 \tau' \big\| F_M' \big\|_{H^{1/2}}^2 d\tau' \right) \lesssim \tau^6 2^{6k} \big(\mathcal{D}_{II} + \mathcal{F}_{II}(\tau) \big).
\end{aligned}$$

Collecting all the bounds proved for the error terms, we conclude the proof. ☐

Chapter 4

The Scattering Map

9 Estimates from $\{v=0\}$ to $\{v=-u\}$

In this section we prove optimal estimates on the smooth spacetime $(\mathcal{M},g)$ obtained in Theorem 3.1 in terms of the asymptotic data at $\{u=-1,\ v=0\}$. Using the ambient metric construction, in the original $(n+1)$-dimensional formulation these correspond to proving estimates at finite times in terms of the asymptotic data at $\mathcal{I}^-$. This section is based on [Cic24, Section 8].

We first introduce the notion of asymptotic data set at $\{u=-1,\ v=0\}$:

Definition 9.1. *Let $\big(\slashed{g}_0,h\big)$ be smooth straight initial data at $\{u=-1,\ v=0\}$. We define the corresponding asymptotic data set at $\{u=-1,\ v=0\}$ by:*

$$\Sigma\big(\slashed{g}_0,h\big):=\Bigg\{\slashed{g}_0,\Big\{\nabla_4^l\psi:\ 0\leq l\leq\frac{n-4}{2}\Big\},\Big\{\nabla_4^l\Psi^G:\ 0\leq l\leq\frac{n-4}{2}\Big\},$$
$$\Big\{\nabla_4^l\alpha:\ 0\leq l\leq\frac{n-6}{2}\Big\},\mathcal{O},\mathfrak{h}\Bigg\},$$

where we have that for all admissible l the tensors $\nabla_4^l\psi,\nabla_4^l\Psi^G,\nabla_4^l\alpha$, and $\mathcal{O}$ are defined in terms of $\slashed{g}_0$ by the compatibility relations in the Fefferman-Graham expansion as in [RSR18, Proposition 4.3], and we define $\mathfrak{h}=h-2(\log\nabla)\mathcal{O}$. We also denote by $\Sigma_{\text{Minkowski}}=\Sigma\big(\slashed{g}_{S^n},0\big)$ the Minkowski data set.

For $M > 0$ large enough, we define the asymptotic data norm of order M, measuring closeness to the Minkowski data:

$$\Big\|\Sigma\big(\slashed{g}_0, h\big)\Big\|^2_M = \|\slashed{g}^*_0\|^2_{\mathring{H}^{M+1}} + \sum_{k=0}^{1} \|\nabla^{M+1+k}\nabla_4^{\frac{n-4}{2}-k}\psi^*\|^2_{H^{1/2}} + \sum_{l=0}^{\frac{n-4}{2}} \sum_{m=0}^{M+\frac{n-4}{2}-l} \|\nabla_4^l\psi^*\|^2_{H^{m+1}} +$$

$$+\sum_{l=0}^{\frac{n-4}{2}} \sum_{m=0}^{M+\frac{n-4}{2}-l} \|\nabla_4^l\Psi^G\|^2_{H^{m+1}} + \sum_{l=0}^{\frac{n-6}{2}} \sum_{m=0}^{M+\frac{n-4}{2}-l} \|\nabla_4^l\alpha\|^2_{H^{m+1}} + \|\mathcal{O}\|^2_{H^{M+1}} + \|\mathfrak{h}\|^2_{H^{M+1}},$$

where $\mathring{H}^{M+1}$ is the Sobolev space with respect to $\slashed{g}_{S^n}$, and the other Sobolev spaces are defined with respect to $\slashed{g}$.

For every $\epsilon > 0$ small enough, we denote the set of $\epsilon-$small asymptotic data by:

$$B^M_\epsilon\big(\Sigma_{\text{Minkowski}}\big) = \Big\{\Sigma\big(\slashed{g}_0, h\big): \ \big(\slashed{g}_0, h\big) \textit{ smooth straight initial data, } \big\|\Sigma\big\|_M < \epsilon\Big\}.$$

Remark 9.1. *For $\big(\underline{\slashed{g}_0}, \underline{h}\big)$ asymptotic data at $\{u = 0, v = 1\}$, the precise notion of asymptotic data norm is obtained via the transformation $(u, v) \to (-v, -u)$ in the above definition. In particular, we notice that we replace $\Psi^G, \alpha, \mathcal{O}, \mathfrak{h}$ by $\underline{\Psi}^G, \underline{\alpha}, \underline{\mathcal{O}}, \underline{\mathfrak{h}}$, and all ∇_4 derivatives by ∇_3 derivatives. However, in view of the compatibility relations, the quantities in $\Sigma\big(\underline{\slashed{g}_0}, \underline{h}\big)$ are expressed in terms of $\big(\underline{\slashed{g}_0}, \underline{h}\big)$ using the same formulas as the ones satisfied by $\Sigma\big(\slashed{g}_0, h\big)$ in terms of $\big(\slashed{g}_0, h\big)$.*

The main result of this section is the following estimate:

Theorem 9.1. *For any $M > 0$ large enough and $\epsilon > 0$ small enough we consider the smooth straight initial data $\big(\slashed{g}_0, h\big)$ such that $\Sigma\big(\slashed{g}_0, h\big) \in B^M_\epsilon\big(\Sigma_{\text{Minkowski}}\big)$. The smooth spacetime $(\mathcal{M}, g)$ obtained in Theorem 3.1 with asymptotic initial data given by $\big(\slashed{g}_0, h\big)$ satisfies the following estimate on $S_{(-1,1)}$:*

$$\Xi^2_M := \sum_{i+j=0}^{\frac{n-4}{2}} \sum_{m=0}^{M+\frac{n-4}{2}-i-j} \big\|\nabla_3^i\nabla_4^j\Psi\big\|^2_{H^{m+1}} + \sum_{i+j=0}^{\frac{n-4}{2}} \big\|\nabla^M\nabla_3^i\nabla_4^j\Psi\big\|^2_{H^{3/2}}$$

$$+ \sum_{i+j=0}^{\frac{n-2}{2}} \big\|\nabla^M\nabla_3^i\nabla_4^j\Psi\big\|^2_{H^{1/2}} + \big\|\slashed{g}^*\big\|^2_{H^{M+1}} + \sum_{k=0}^{1} \sum_{i+j=\frac{n-4}{2}-k} \big\|\nabla^{M+1+k}\nabla_3^i\nabla_4^j\psi^*\big\|^2_{H^{1/2}}$$

$$+ \sum_{i+j=0}^{\frac{n-4}{2}} \sum_{m=0}^{M+\frac{n-4}{2}-i-j} \big\|\nabla_3^i\nabla_4^j\psi^*\big\|^2_{H^{m+1}} \lesssim \Big\|\Sigma\big(\slashed{g}_0, h\big)\Big\|^2_M.$$

We remark that by self-similarity, this result follows from the corresponding estimates along $\{u=-1\}$, where we replace all ∇_3 derivatives by ∇_4. We restrict our analysis to $\{u=-1\}$ for the rest of the section.

We briefly outline the structure of the argument for the rest of the section. The idea of the proof is to define suitable norms in Section 9.1 for the curvature components and Ricci coefficients, which we then estimate one at a time in Sections 9.2 and 9.3. Finally, we combine our estimates to complete the proof of Theorem 9.1 in Section 9.4.

We further outline the steps of our proof in more detail to assist the reader throughout the section. We define the top order curvature norms $\mathcal{T}$ in (9.1) and $\mathcal{S}$ in (9.2). We estimate these in Corollary 9.1 in Section 9.2 as a consequence of Theorem 7.1. We define the lower order curvature norms $\mathcal{L}$ in (9.3) and $\mathcal{M}_l$ in (9.4). We estimate these in Propositions 9.1 and 9.2 in Section 9.2. We define the Ricci coefficients norm $\mathcal{R}$ in (9.5) and estimate it in Proposition 9.3 in Section 9.2. The estimates outlined so far introduce certain nonlinear error terms. We estimate these in Propositions 9.4 and 9.5 in Section 9.3. A key part in controlling the error terms is the estimate (9.10) for the lower order pointwise norms $\mathcal{P}$ in (9.6) and $\mathcal{SP}$ in (9.7), which follows from Section 3. In Section 9.4 we combine all these estimates to complete the proof of Theorem 9.1.

9.1 Norms

Following the above outline, we define the following norms on $\big\{u=-1,\ 0\leq v\leq 1\big\}:$

- Top order energy $\mathcal{T}=\mathcal{T}(-1,v)$:

$$\mathcal{T}=v^2\big\|\nabla_4\nabla^M\nabla_4^{\frac{n-4}{2}}\Psi\big\|^2_{H^{1/2}}+v\big\|\nabla^M\nabla_4^{\frac{n-4}{2}}\Psi\big\|^2_{H^{3/2}}+\big\|\nabla_4^{\frac{n-4}{2}}\Psi^G\big\|^2_{H^{M+1}}. \tag{9.1}$$

- Mildly singular top order energy $\mathcal{S}=\mathcal{S}(-1,v)$:

$$\mathcal{S}=\big\|\nabla_4^{\frac{n-4}{2}}\alpha\big\|^2_{H^{M+1}}. \tag{9.2}$$

- Lower order energy $\mathcal{L}=\mathcal{L}(-1,v)$:

$$\mathcal{L}=\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}v\big\|\nabla_4^{l+1}\Psi\big\|^2_{H^m}+\big\|\nabla_4^{l}\Psi\big\|^2_{H^{m+1}}. \tag{9.3}$$

- Fractional lower order energy $\mathcal{M}_l=\mathcal{M}_l(-1,v)$ for any $0\leq l\leq\frac{n-6}{2}$:

$$\mathcal{M}_l=\big\|\nabla^M\nabla_4^l\Psi\big\|^2_{H^{5/2}}. \tag{9.4}$$

- Ricci coefficients norm $\mathcal{R} = \mathcal{R}(-1, v)$:

$$\mathcal{R} = \sum_{k=0}^{1} \big\| \nabla^{M+1+k} \nabla_4^{\frac{n-4}{2}-k} \slashed{\psi}^* \big\|^2_{H^{1/2}} + \sum_{l=0}^{\frac{n-4}{2}} \sum_{m=0}^{M+\frac{n-4}{2}-l} \big\| \nabla_4^l \slashed{\psi}^* \big\|^2_{H^{m+1}}. \tag{9.5}$$

- Lower order pointwise norm $\mathcal{P} = \mathcal{P}(-1, v)$ for $N' = \frac{M}{2} + \frac{n}{4}$:

$$\mathcal{P} = \sum_{l=0}^{\frac{n-6}{2}} \sum_{m=0}^{N'} \big\| \nabla^m \nabla_4^l \Psi \big\|^2_{L^\infty} + \sum_{l=0}^{\frac{n-4}{2}} \sum_{m=0}^{N'} \big\| \nabla^m \nabla_4^l \slashed{\psi}^* \big\|^2_{L^\infty}. \tag{9.6}$$

- Mildly singular pointwise norm $\mathcal{SP} = \mathcal{SP}(-1, v)$:

$$\mathcal{SP} = \sum_{m=0}^{N'} \big\| \nabla^m \nabla_4^{\frac{n-4}{2}} \Psi \big\|^2_{L^\infty} + \sum_{m=0}^{N'} \big\| \nabla^m \nabla_4^{\frac{n-2}{2}} \slashed{\psi}^* \big\|^2_{L^\infty}. \tag{9.7}$$

- Initial data norm $\mathcal{D}$:

$$\mathcal{D} = \Big\| \Sigma\big(\slashed{g}_0, h\big) \Big\|^2_{M}. \tag{9.8}$$

9.2 Estimates for the Norms

We start with some preliminary estimates for the pointwise norms $\mathcal{P}$ and $\mathcal{SP}$ which follow from the estimates of Section 3. Since $\big\| \slashed{g}_0^* \big\|_{\mathring{H}^{M+1}} < \epsilon$, we get by (3.48) that:

$$\sum_{m \leq M+1} \big\| \mathcal{L}_\theta^m \slashed{g}_0^* \big\|_{L^2(S^n)} \lesssim \epsilon. \tag{9.9}$$

Additionally, since $\big\| \mathcal{O} \big\|_{H^{M+1}} < \epsilon$ we also have that $\big\| (\log \nabla) \mathcal{O} \big\|_{H^M} \lesssim \epsilon$. Thus, $\big\| \mathfrak{h} \big\|_{H^{M+1}} < \epsilon$ implies that $\big\| h \big\|_{H^M} \lesssim \epsilon$, so the smallness condition (3.1) holds. As a result, we can apply Propositions 3.1 and 3.2 for $N_1, N_2 = N' = \frac{M}{2} + \frac{n}{4}$ to get that:

$$\mathcal{P} \leq \epsilon, \ \mathcal{SP} \leq \epsilon \big(1 + |\log(v)|^2 \big). \tag{9.10}$$

The bounds for the top order energies $\mathcal{T}$ and $\mathcal{S}$ rely on the refined estimates in Theorem 7.1. Once we established these, we can bound the remaining norms $\mathcal{L}, \mathcal{M}$, and $\mathcal{R}$ using standard estimates.

As a consequence of Theorem 7.1, we obtain the following estimates for the top order energies:

Corollary 9.1. *The top order energy $\mathcal{T}$ and the mildly singular top order energy $\mathcal{S}$ satisfy the estimates for $0 \leq v \leq 1$:*

$$\mathcal{T} \lesssim \mathcal{D} + \sum_{m=0}^{M} \int_0^v v'^{-\frac{1}{2}} \Big\| Err^{\Psi}_{m, \frac{n-4}{2}} \Big\|^2_{L^2} dv' + \int_0^v \Big\| Err^{\Psi}_{M, \frac{n-4}{2}} \Big\|^2_{H^{1/2}} dv', \tag{9.11}$$

$$\mathcal{S} \lesssim \big(1+|\log v|^2\big)\mathcal{D} + \sum_{m=0}^{M}\int_0^v v'^{-\frac{1}{2}}\Big\|Err^{\Psi}_{m,\frac{n-4}{2}}\Big\|^2_{L^2}dv' + \int_0^v \Big\|Err^{\Psi}_{M,\frac{n-4}{2}}\Big\|^2_{H^{1/2}}dv'. \tag{9.12}$$

Proof. We recall that according to Section 5 and Remark 5.4:

$$\Phi_0 = \nabla_4^{\frac{n-4}{2}}\alpha,\ \Phi_i = \nabla_4^{\frac{n-4}{2}}\Psi^G,\ F^0_m = Err^{\Psi}_{m,\frac{n-4}{2}},\ F^i_m = Err^{\Psi}_{m,\frac{n-4}{2}}$$

satisfy the first model system as defined in (5.13) and also [Cic26, Definition 1.1]. The bounds on the background $\big(\mathcal{M},g\big)$ required in [Cic26, Theorem 1.1] follow by Theorem 3.1. We also have that $\big\|\not{R}i\not{e}m(\not{g}_0)\big\|_{H^M}\lesssim\mathcal{D}\lesssim 1$, so we can apply Theorem 7.1 with an implicit constant depending only on M. We change coordinates to $v=\tau^2$ to obtain:

$$v^2\big\|\nabla_4\nabla^M\nabla_4^{\frac{n-4}{2}}\alpha\big\|^2_{H^{1/2}} + v\big\|\nabla^M\nabla_4^{\frac{n-4}{2}}\alpha\big\|^2_{H^{3/2}} \lesssim \big\|\mathcal{O}\big\|^2_{H^{M+1}} + \big\|\mathfrak{h}\big\|^2_{H^{M+1}}$$

$$+\big\|\nabla_4^{\frac{n-4}{2}}\Psi^G_0\big\|^2_{H^{M+1}} + \sum_{m=0}^{M}\int_0^v v'^{-\frac{1}{2}}\Big\|Err^{\Psi}_{m,\frac{n-4}{2}}\Big\|^2_{L^2}dv' + \int_0^v \Big\|Err^{\Psi}_{M,\frac{n-4}{2}}\Big\|^2_{H^{1/2}}dv'.$$

$$\big\|\nabla_4^{\frac{n-4}{2}}\alpha\big\|^2_{H^{M+1}} \lesssim \big(1+|\log v|^2\big)\big\|\mathcal{O}\big\|^2_{H^{M+1}} + \big\|\mathfrak{h}\big\|^2_{H^{M+1}} + \big\|\nabla_4^{\frac{n-4}{2}}\Psi^G_0\big\|^2_{H^{M+1}} +$$

$$+\sum_{m=0}^{M}\int_0^v v'^{-\frac{1}{2}}\Big\|Err^{\Psi}_{m,\frac{n-4}{2}}\Big\|^2_{L^2}dv' + \int_0^v \Big\|Err^{\Psi}_{M,\frac{n-4}{2}}\Big\|^2_{H^{1/2}}dv'.$$

$$v^{\frac{3}{2}}\big\|\nabla_4\nabla^M\nabla_4^{\frac{n-4}{2}}\Psi^G\big\|^2_{H^{1/2}} + \sqrt{v}\big\|\nabla^M\nabla_4^{\frac{n-4}{2}}\Psi^G\big\|^2_{H^{3/2}} + \big\|\nabla_4^{\frac{n-4}{2}}\Psi^G\big\|^2_{H^{M+1}} \lesssim \big\|\mathcal{O}\big\|^2_{H^{M+1}} + \big\|\mathfrak{h}\big\|^2_{H^{M+1}}$$

$$+\big\|\nabla_4^{\frac{n-4}{2}}\Psi^G_0\big\|^2_{H^{M+1}} + \sum_{m=0}^{M}\int_0^v v'^{-\frac{1}{2}}\Big\|Err^{\Psi}_{m,\frac{n-4}{2}}\Big\|^2_{L^2}dv' + \int_0^v \Big\|Err^{\Psi}_{M,\frac{n-4}{2}}\Big\|^2_{H^{1/2}}dv'.$$

Using the notation (9.8) for the initial data norm $\mathcal{D}$, we obtain the conclusion. □

We prove the following result for the lower order energy $\mathcal{L}$ defined in (9.3):

Proposition 9.1. *The lower order energy* $\mathcal{L}$ *satisfies the estimate for* $0\leq v\leq 1$ *:*

$$\mathcal{L} \lesssim \mathcal{D} + \epsilon\mathcal{R} + \int_0^v\big(\mathcal{R}+\mathcal{T}+\mathcal{S}\big)dv' + \sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_0^v\Big\|Err^{\Psi}_{m,l}\Big\|^2_{L^2}dv'. \tag{9.13}$$

Proof. We recall that by (5.11), for every $0\leq l\leq\frac{n-6}{2}$ and $0\leq m\leq M+\frac{n-4}{2}-l$ we have that all curvature components Ψ satisfy the commuted equation:

$$v\nabla_4^2\nabla^m\nabla_4^l\Psi + \bigg(3+l-\frac{n}{2}\bigg)\nabla_4\nabla^m\nabla_4^l\Psi - 4\Delta\nabla^m\nabla_4^l\Psi = \sum_{\mu}\psi\nabla^{m+1}\nabla_4^l\Psi_{\mu} + Err^{\Psi}_{ml}.$$

We contract each equation with $\nabla_4\nabla^m\nabla_4^l\Psi$, then sum over all m, l in the admissible range and all curvature components Ψ to obtain:

$$\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l} v\big\|\nabla_4\nabla^m\nabla_4^l\Psi\big\|_{L^2}^2+\big\|\nabla\nabla_4^l\Psi\big\|_{H^m}^2\lesssim\mathcal{D}+\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_0^v\big\|\nabla_4\nabla^m\nabla_4^l\Psi\big\|_{L^2}^2+$$

$$+\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_0^v\big\|[\nabla,\nabla_4]\nabla^m\nabla_4^l\Psi\big\|_{L^2}^2+\big\|\nabla^{m+1}\nabla_4^l\Psi\big\|_{L^2}^2+\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_0^v\big\|Err_{ml}^{\Psi}\big\|_{L^2}^2.$$

We also have the estimate:

$$\big\|\nabla_4^l\Psi\big\|_{L^2}^2\lesssim\mathcal{D}+\int_0^v\big\|\nabla_4^{l+1}\Psi\big\|_{L^2}^2\lesssim\ldots\lesssim\mathcal{D}+\int_0^v\big\|\nabla_4^{\frac{n-4}{2}}\Psi\big\|_{L^2}^2\lesssim\mathcal{D}+\int_0^v\mathcal{T}+\mathcal{S}.$$

We use the commutation formulas in Lemma 2.7 and Gronwall to obtain:

$$\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l} v\big\|\nabla_4\nabla^m\nabla_4^l\Psi\big\|_{L^2}^2+\big\|\nabla_4^l\Psi\big\|_{H^{m+1}}^2\lesssim$$

$$\lesssim\mathcal{D}+\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_0^v\mathcal{T}+\mathcal{S}+\big\|\nabla^m\nabla_4^{l+1}\Psi\big\|_{L^2}^2+\big\|\psi\nabla_4^l\Psi\big\|_{H^m}^2+\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_0^v\big\|Err_{ml}^{\Psi}\big\|_{L^2}^2.$$

Once again, we use Gronwall and bound the other terms using $\mathcal{T},\mathcal{S}$, and $\mathcal{R}$:

$$\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l} v\big\|\nabla_4\nabla^m\nabla_4^l\Psi\big\|_{L^2}^2+\big\|\nabla_4^l\Psi\big\|_{H^{m+1}}^2\lesssim$$

$$\lesssim\mathcal{D}+\int_0^v\big(\mathcal{T}+\mathcal{S}+\mathcal{R}\big)+\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_0^v\big\|Err_{ml}^{\Psi}\big\|_{L^2}^2.$$

Using this bound and the lower order pointwise estimate (9.10), we get:

$$\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l} v\big\|\psi\nabla_4^l\Psi\big\|_{H^m}^2\lesssim$$

$$\lesssim\sum_{m=N'}^{M+\frac{n-4}{2}}\big\|\psi^*\big\|_{H^m}^2\cdot\sum_{l=0}^{\frac{n-6}{2}}\big\|\nabla_4^l\Psi\big\|_{W^{N',\infty}}^2+\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\big\|\nabla_4^l\Psi\big\|_{H^m}^2\cdot\big\|\psi\big\|_{W^{N',\infty}}^2$$

$$\lesssim\epsilon\mathcal{R}+\mathcal{D}+\int_0^v\big(\mathcal{T}+\mathcal{S}+\mathcal{R}\big)+\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_0^v\big\|Err_{ml}^{\Psi}\big\|_{L^2}^2.$$

We obtain the conclusion by using the commutation formulas in Lemma 2.7:

$$\mathcal{L}\lesssim\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l} v\big\|\psi\nabla_4^l\Psi\big\|_{H^m}^2+\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l} v\big\|\nabla_4\nabla^m\nabla_4^l\Psi\big\|_{L^2}^2+\big\|\nabla_4^l\Psi\big\|_{H^{m+1}}^2.$$

□

We prove the following result for the fractional lower order energy defined in (9.4):

Proposition 9.2. *The fractional lower order energy* $\mathcal{M}_l$ *satisfies the estimate for any* $0 \leq l \leq \frac{n-6}{2}$ *and* $0 \leq v \leq 1$ *:*

$$\mathcal{M}_l \lesssim \mathcal{T} + \mathcal{S} + \mathcal{L} + \mathcal{R} + \left\| Err^{\Psi}_{M,l} \right\|^2_{H^{1/2}}. \tag{9.14}$$

Proof. We have the estimate for any $0 \leq l \leq \frac{n-6}{2}$:

$$\left\| \nabla^M \nabla_4^l \Psi \right\|^2_{H^{5/2}} \lesssim \left\| \nabla_4^l \Psi \right\|^2_{H^{M+2}} + \left\| \Delta \nabla^M \nabla_4^l \Psi \right\|^2_{H^{1/2}}$$

$$\lesssim \mathcal{L} + \left\| v \nabla_4^2 \nabla^M \nabla_4^l \Psi \right\|^2_{H^{1/2}} + \left\| \nabla_4 \nabla^M \nabla_4^l \Psi \right\|^2_{H^{1/2}} + \left\| \nabla^{M+1} \nabla_4^l \Psi \right\|^2_{H^{1/2}} + \left\| Err^{\Psi}_{M,l} \right\|^2_{H^{1/2}}$$

$$\lesssim \mathcal{L} + \left\| v \nabla_4 \nabla^M \nabla_4^{l+1} \Psi \right\|^2_{H^{\frac{1}{2}}} + \left\| v \nabla_4 \nabla^M \big(\psi \nabla_4^l \Psi \big) \right\|^2_{H^{\frac{1}{2}}} + \left\| \nabla^M \nabla_4^{l+1} \Psi \right\|^2_{H^{\frac{1}{2}}}$$

$$+ \left\| \nabla^M \big(\psi \nabla_4^l \Psi \big) \right\|^2_{H^{\frac{1}{2}}} + \left\| Err^{\Psi}_{M,l} \right\|^2_{H^{\frac{1}{2}}}$$

$$\lesssim \mathcal{L} + \mathcal{T} + \mathcal{S} + \mathcal{R} + \left\| v \nabla_4 \nabla^M \nabla_4^{l+1} \Psi \right\|^2_{H^{1/2}} + \left\| v \nabla_4 \nabla^M \big(\psi \nabla_4^l \Psi \big) \right\|^2_{H^{1/2}} + \left\| Err^{\Psi}_{M,l} \right\|^2_{H^{1/2}}.$$

For $l = \frac{n-6}{2}$, we have the bound by (9.1):

$$\left\| v \nabla_4 \nabla^M \nabla_4^{\frac{n-4}{2}} \Psi \right\|^2_{H^{1/2}} \lesssim \mathcal{T}.$$

For $0 \leq l \leq \frac{n-8}{2}$, we have:

$$\left\| v \nabla_4 \nabla^M \nabla_4^{l+1} \Psi \right\|^2_{H^{1/2}} \lesssim \left\| v \nabla^M \nabla_4^{l+2} \Psi \right\|^2_{H^{1/2}} + \left\| v \nabla^M \big(\psi \nabla_4^{l+1} \Psi \big) \right\|^2_{H^{1/2}} \lesssim \mathcal{T} + \mathcal{S} + \mathcal{L} + \mathcal{R}.$$

So far, we proved that:

$$\mathcal{M}_l \lesssim \mathcal{T} + \mathcal{S} + \mathcal{L} + \mathcal{R} + \left\| v \nabla_4 \nabla^M \big(\psi \nabla_4^l \Psi \big) \right\|^2_{H^{1/2}} + \left\| Err^{\Psi}_{M,l} \right\|^2_{H^{1/2}}.$$

We can conclude since we also have the bounds using the null structure equations in Proposition 2.4:

$$\left\| v \nabla_4 \nabla^M \big(\psi \nabla_4^l \Psi \big) \right\|^2_{H^{1/2}} \lesssim \left\| v \nabla^M \nabla_4 \big(\psi \nabla_4^l \Psi \big) \right\|^2_{H^{1/2}} + \left\| v \nabla^M \big(\psi^2 \nabla_4^l \Psi \big) \right\|^2_{H^{1/2}} \lesssim$$

$$\lesssim \left\| v \nabla^M \big(\psi \nabla_4^{l+1} \Psi \big) \right\|^2_{H^{1/2}} + \left\| v \nabla^M \big((\Psi + \psi\psi) \nabla_4^l \Psi \big) \right\|^2_{H^{1/2}} + \mathcal{L} + \mathcal{R} \lesssim \mathcal{T} + \mathcal{S} + \mathcal{L} + \mathcal{R}.$$

□

We prove the following result for the Ricci coefficients norm $\mathcal{R}$ defined in (9.5):

Proposition 9.3. *The Ricci coefficients norm $\mathcal{R}$ satisfies the estimate for any $0 \leq v \leq 1$:*

$$\mathcal{R} \lesssim \mathcal{D} + \int_0^v \big((v')^{-1/2}\mathcal{T} + \mathcal{S} + \mathcal{L} + \mathcal{R} + (v')^{1/2}\mathcal{M}_{\frac{n-6}{2}}\big)dv'. \tag{9.15}$$

Proof. For every $0 \leq l \leq \frac{n-4}{2}$ and $0 \leq m \leq M + \frac{n-4}{2} - l$, the Ricci coefficients ψ^* satisfy the commuted equation:

$$\nabla_4 \nabla^{m+1} \nabla_4^l \psi^* = \nabla^{m+1}\nabla_4^l \Psi + \nabla^{m+1}\nabla_4^l\big(\psi\psi^*\big) + [\nabla_4, \nabla^{m+1}]\nabla_4^l \psi^*. \tag{9.16}$$

We obtain the estimate:

$$\sum_{l=0}^{\frac{n-4}{2}} \sum_{m=0}^{M+\frac{n-4}{2}-l} \big\|\nabla_4^l \psi^*\big\|^2_{H^{m+1}} \lesssim \mathcal{D} +$$

$$+ \sum_{l=0}^{\frac{n-4}{2}} \sum_{m=0}^{M+\frac{n-4}{2}-l} \int_0^v \big\|\nabla_4^l \Psi\big\|^2_{H^{m+1}} + \big\|\nabla_4^l\big(\psi\psi^*\big)\big\|^2_{H^{m+1}} + \big\|\psi\nabla_4^l\psi^*\big\|^2_{H^{m+1}}$$

$$\lesssim \mathcal{D} + \int_0^v \big(\mathcal{T} + \mathcal{S} + \mathcal{L} + \mathcal{R}\big)dv'.$$

Using the LP projections in Section 6 and Gronwall, we also obtain the following fractional estimate:

$$\big\|\nabla^{M+1}\nabla_4^{\frac{n-4}{2}}\psi^*\big\|^2_{H^{1/2}} \lesssim \mathcal{D} + \int_0^v \mathcal{T} + \mathcal{S} + \mathcal{L} + \mathcal{R} + \int_0^v (v')^{1/2}\big\|\nabla_4\nabla^{M+1}\nabla_4^{\frac{n-4}{2}}\psi^*\big\|^2_{H^{1/2}}$$

$$\lesssim \mathcal{D} + \int_0^v \mathcal{T} + \mathcal{S} + \mathcal{L} + \mathcal{R} + \sum_{m=0}^{M+1}\int_0^v \big\|\nabla^m\nabla_4^{\frac{n-4}{2}}\big(\psi\psi^*\big)\big\|^2_{H^{1/2}} + (v')^{1/2}\big\|\nabla^m\nabla_4^{\frac{n-4}{2}}\Psi\big\|^2_{H^{1/2}}$$

$$\lesssim \mathcal{D} + \int_0^v \frac{\mathcal{T}}{(v')^{1/2}} + \mathcal{S} + \mathcal{L} + \mathcal{R}.$$

Similarly, we also get the estimate:

$$\big\|\nabla^{M+2}\nabla_4^{\frac{n-6}{2}}\psi^*\big\|^2_{H^{1/2}} \lesssim \mathcal{D} + \int_0^v \mathcal{T} + \mathcal{S} + \mathcal{L} + \mathcal{R} + \int_0^v (v')^{1/2}\big\|\nabla_4\nabla^{M+2}\nabla_4^{\frac{n-6}{2}}\psi^*\big\|^2_{H^{1/2}}$$

$$\lesssim \mathcal{D} + \int_0^v \mathcal{T} + \mathcal{S} + \mathcal{L} + \mathcal{R} + \sum_{m=0}^{M+2}\int_0^v \big\|\nabla^m\nabla_4^{\frac{n-6}{2}}\big(\psi\psi^*\big)\big\|^2_{H^{1/2}} + (v')^{1/2}\big\|\nabla^m\nabla_4^{\frac{n-6}{2}}\Psi\big\|^2_{H^{1/2}}$$

$$\lesssim \mathcal{D} + \int_0^v \mathcal{T} + \mathcal{S} + \mathcal{L} + \mathcal{R} + (v')^{1/2}\mathcal{M}_{\frac{n-6}{2}}.$$

□

9.3 Error Term Estimates

In this section, we bound the error terms that appear in the above estimates in terms of the norms introduced in Section 9.1. We notice that similarly to Section 3, these do not create significant difficulties since we commuted the equations with a high number of angular derivatives. We recall that for any $0 \leq l \leq \frac{n-4}{2}$ and $0 \leq m \leq M + \frac{n-4}{2} - l$ we have:

$$Err^{\Psi}_{ml} = v\mathcal{F}_{(m)(l+1)(l+m+1)}(\Psi) + v\mathcal{F}_{(m+1)(l)(l+m+1)}(\Psi) + \mathcal{F}_{(2+m)(l-1)(l+m+1)}(\Psi) + \mathcal{F}_{(m)(l)(m+l)}(\Psi)$$
$$+\mathcal{F}^{lot}_{(m+1)(l)(m+l+1)}(\Psi) + \nabla^m\nabla_4^l\big(\Psi \cdot \Psi^G\big)$$
$$+ \sum_{i+2j=m} \nabla^i\big(Ri\not{e}m^{j+1} \cdot \nabla_4^l\Psi\big) + \nabla^m\nabla_4^l\nabla\big(\psi\psi\big) + \nabla^m\nabla_4^l\nabla\big(\psi\psi\psi\big).$$

We also recall the error term notation in Definition 2.4:

$$\mathcal{F}_{mlp}(\Psi) = \sum_{\substack{i+j+k\leq p\\ i\leq l, k\leq m}} \nabla^k\nabla_4^i\big(\psi^{j+1}\Psi\big).$$

We start with a general result for the error terms $\mathcal{F}_{mlp}$:

Lemma 9.1. *For any* $0 \leq s \leq 1,\ l \leq \frac{n-2}{2},\ m \leq M + \frac{n}{2},\ p = m + l$ *we have the estimate:*

$$\big\|\mathcal{F}_{mlp}(\Psi)\big\|^2_{H^s} \lesssim \big(1 + |\log(v)|^2\big) \cdot \sum_{i\leq l}\sum_{k\leq m} \big\|\nabla^k\nabla_4^i\Psi\big\|^2_{H^s} + \big\|\nabla^k\nabla_4^i\psi^*\big\|^2_{H^s}. \tag{9.17}$$

Similarly, for any $0 \leq s \leq 1,\ l \leq \frac{n-4}{2},\ m \leq M + \frac{n}{2},\ p = m + l$ *we have the estimate:*

$$\big\|\mathcal{F}_{mlp}(\Psi)\big\|^2_{H^s} \lesssim \sum_{i\leq l}\sum_{k\leq m} \big\|\nabla^k\nabla_4^i\Psi\big\|^2_{H^s} + \big\|\nabla^k\nabla_4^i\psi^*\big\|^2_{H^s}. \tag{9.18}$$

Proof. Using the definition of $\mathcal{F}_{mlp}$, we get that for all $0 \leq s \leq 1,\ l \leq \frac{n-2}{2},\ m \leq M + \frac{n}{2},\ p = m+l$:

$$\big\|\mathcal{F}_{mlp}(\Psi)\big\|^2_{H^s} \lesssim \sum_{\substack{i+j+k\leq p\\ i\leq l, k\leq m}} \big\|\nabla^k\nabla_4^i\big(\psi^{j+1}\Psi\big)\big\|^2_{H^s} \lesssim \sum_{\substack{|i|+j+|k|\leq p\\ |i|\leq l, |k|\leq m}} \Big\|\nabla^{k_0}\nabla_4^{i_0}\Psi\prod_{q=1}^{j+1}\nabla^{k_q}\nabla_4^{i_q}\psi\Big\|^2_{H^s}.$$

We use the fact that $\nabla\nabla_4^i\psi = \nabla\nabla_4^i\psi^*$ to get:

$$\big\|\mathcal{F}_{mlp}(\Psi)\big\|^2_{H^s} \lesssim \sum_{\substack{|i|+j+|k|\leq p\\ |i|\leq l, |k|\leq m}} \Big\|\nabla^{k_0}\nabla_4^{i_0}\Psi \cdot \nabla^{k_1}\nabla_4^{i_1}\psi^*\prod_{q=2}^{j+1}\nabla^{k_q}\nabla_4^{i_q}\psi\Big\|^2_{H^s}$$
$$+ \sum_{\substack{|i|+j+|k|\leq p\\ |i|\leq l, |k|\leq m}} \Big\|\nabla^{k}\nabla_4^{i_0}\Psi\prod_{q=1}^{j+1}\nabla_4^{i_q}\psi\Big\|^2_{H^s}.$$

The second term can be bounded using (9.10) for the $\mathcal{P}$ and $\mathcal{SP}$ pointwise norms:

$$\sum_{\substack{|i|+j+|k|\leq p\\|i|\leq l,|k|\leq m}}\Big\|\nabla^k\nabla_4^{i_0}\Psi\prod_{q=1}^{j+1}\nabla_4^{i_q}\psi\Big\|_{H^s}^2\lesssim\big(1+|\log(v)|^2\big)\cdot\sum_{i\leq l}\sum_{k\leq m}\big\|\nabla^k\nabla_4^i\Psi\big\|_{H^s}^2.$$

For the first term, we can assume that $|k_1|=\max(|k_1|,\ldots,|k_{j+1}|)$. In particular, we have $|k_q|<1+\frac{M}{2}+\frac{n}{4}$ for all $2\leq q\leq j+1$, so we can control these factors using the lower order pointwise norm $\mathcal{P}$ and the mildly singular pointwise norm $\mathcal{SP}$. Thus, we get that the first term is bounded by:

$$\big(1+|\log(v)|^2\big)\cdot\sum_{\substack{|i|+j+|k|\leq p\\|i|\leq l,|k|\leq m}}\Big\|\nabla^{k_0}\nabla_4^{i_0}\Psi\cdot\nabla^{k_{q_1}}\nabla_4^{i_{q_1}}\psi^*\Big\|_{H^s}^2\lesssim$$

$$\lesssim\big(1+|\log(v)|^2\big)\cdot\sum_{i\leq l}\sum_{k\leq m}\big\|\nabla^k\nabla_4^i\Psi\big\|_{H^s}^2+\big\|\nabla^k\nabla_4^i\psi^*\big\|_{H^s}^2.$$

This completes the proof of (9.17). We remark that we can prove (9.18) by following the exact same steps, but we only use the lower order pointwise norm $\mathcal{P}$ when $l\leq\frac{n-4}{2}$. $\square$

Proposition 9.4. *For any $0\leq l\leq\frac{n-4}{2}$, the error terms $Err_{M,l}^{\Psi}$ satisfy the estimate:*

$$\big\|Err_{M,l}^{\Psi}\big\|_{H^{1/2}}^2\lesssim\big(1+|\log(v)|^2\big)\big(\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}\big). \tag{9.19}$$

Proof. The proof of (9.19) will follow from the fractional lower order energy estimates (9.14), once we prove the claim:

$$\big\|Err_{M,l}^{\Psi}\big\|_{H^{1/2}}^2\lesssim\big(1+|\log(v)|^2\big)\big(\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}\big)+\mathcal{M}_{l-1}. \tag{9.20}$$

Using (9.17)-(9.18), the Bianchi equations in Proposition 2.6, and the null structure equations in Proposition 2.4, we obtain:

$$\big\|v\mathcal{F}_{(M)(l+1)(l+M+1)}(\Psi)\big\|_{H^{1/2}}^2\lesssim\big(1+|\log(v)|^2\big)\cdot\sum_{i\leq l+1}\sum_{k\leq M}\big\|v\nabla^k\nabla_4^i\Psi\big\|_{H^{1/2}}^2+\big\|v\nabla^k\nabla_4^i\psi^*\big\|_{H^{1/2}}^2$$

$$\lesssim\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}+\big(1+|\log(v)|^2\big)\cdot\Big[\sum_{k\leq M}\big\|v\nabla^k\nabla_4^{\frac{n-2}{2}}\psi^*\big\|_{H^{1/2}}^2+\big\|v\nabla^M\nabla_4^{\frac{n-2}{2}}\Psi\big\|_{H^{1/2}}^2+\big\|v\nabla_4^{\frac{n-2}{2}}\Psi\big\|_{H^1}^2\Big]$$

$$\lesssim\big(1+|\log(v)|^2\big)\bigg(\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}+\sum_{k\leq M}\big\|v\nabla^k\nabla_4^{\frac{n-4}{2}}\big(\Psi+\psi\psi^*\big)\big\|_{H^{1/2}}^2+\big\|\nabla_4^{\frac{n-4}{2}}\big(\nabla\Psi+\psi\Psi+\Psi\big)\big\|_{H^1}^2\bigg)$$

$$\lesssim\big(1+|\log(v)|^2\big)\big(\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}\big).$$

Similarly, we use (9.17)-(9.18) to get that:

$$\begin{aligned}\big\|v\mathcal{F}_{(M+1)(l)(l+M+1)}(\Psi)\big\|^2_{H^{1/2}} &\lesssim \sum_{i\leq l}\sum_{k\leq M+1}\big\|v\nabla^k\nabla_4^i\Psi\big\|^2_{H^{1/2}}+\big\|v\nabla^k\nabla_4^i\psi^*\big\|^2_{H^{1/2}}\\ &\lesssim \mathcal{L}+\mathcal{R}+\sum_{k\leq M}\big\|v\nabla^k\nabla_4^{\frac{n-4}{2}}\psi^*\big\|^2_{H^{3/2}}+\big\|v\nabla^k\nabla_4^{\frac{n-4}{2}}\Psi\big\|^2_{H^{3/2}}\lesssim\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}.\end{aligned}$$

We use the fractional lower order energy (9.4) and (9.18):

$$\begin{aligned}\big\|\mathcal{F}_{(M+2)(l-1)(l+M+1)}(\Psi)\big\|^2_{H^{1/2}} &\lesssim \sum_{i\leq l-1}\sum_{k\leq M+2}\big\|\nabla^k\nabla_4^i\Psi\big\|^2_{H^{1/2}}+\big\|\nabla^k\nabla_4^i\psi^*\big\|^2_{H^{1/2}}\\ &\lesssim \mathcal{L}+\mathcal{R}+\sum_{k\leq M}\big\|\nabla^k\nabla_4^{l-1}\psi^*\big\|^2_{H^{5/2}}+\sum_{k\leq M}\big\|\nabla^k\nabla_4^{l-1}\Psi\big\|^2_{H^{5/2}}\lesssim\mathcal{L}+\mathcal{R}+\mathcal{M}_{l-1}.\end{aligned}$$

Next, we have by (9.18):

$$\big\|\mathcal{F}_{(M)(l)(l+M)}(\Psi)\big\|^2_{H^{1/2}}\lesssim\sum_{i\leq l}\sum_{k\leq M}\big\|\nabla^k\nabla_4^i\Psi\big\|^2_{H^{1/2}}+\big\|\nabla^k\nabla_4^i\psi^*\big\|^2_{H^{1/2}}\lesssim\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}. \tag{9.21}$$

We can prove a similar result to the Lemma 9.1 for the error terms $\mathcal{F}^{lot}$, and we obtain:

$$\begin{aligned}\big\|\mathcal{F}^{lot}_{(M+1)(l)(l+M+1)}(\Psi)\big\|^2_{H^{\frac12}} &\lesssim \sum_{\substack{i\leq l\\ k\leq M}}\big\|\nabla^k\nabla_4^i\Psi\big\|^2_{H^{\frac12}}+\sum_{\substack{i\leq l-1\\ k\leq M+1}}\big\|\nabla^k\nabla_4^i\Psi\big\|^2_{H^{\frac12}}+\sum_{\substack{i\leq l\\ k\leq M+1}}\big\|\nabla^k\nabla_4^i\psi^*\big\|^2_{H^{\frac12}}\\ &\lesssim\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}.\end{aligned}$$

We also use the definitions of the curvature norms (9.1)-(9.3) to get:

$$\big\|\nabla^M\nabla_4^l\big(\Psi\cdot\Psi^G\big)\big\|^2_{H^{1/2}}\lesssim\big(1+|\log(v)|^2\big)\big(\mathcal{T}+\mathcal{S}+\mathcal{L}\big). \tag{9.22}$$

Using (9.22), (9.21), and the formula $Ri\not{e}m=\Psi+\psi\psi$ (implied by the constraint equation (2.19)), we also obtain:

$$\sum_{i+2j=M}\big\|\nabla^i\big(Ri\not{e}m^{j+1}\cdot\nabla_4^l\Psi\big)\big\|^2_{H^{1/2}}\lesssim\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}.$$

Finally, we use the commutation formula in Lemma 2.5 to get:

$$\big\|\nabla^M\nabla_4^l\nabla\big(\psi\psi\psi\big)\big\|^2_{H^{1/2}}\lesssim\mathcal{R}+\sum_{k\leq M+1}\sum_{l\leq\frac{n-4}{2}}\big\|\nabla^k\nabla_4^l\psi^*\big\|^2_{H^{1/2}}\lesssim\mathcal{R}.$$

Combining the estimates proved so far, we establish (9.20). Using (9.14), we get:

$$\begin{aligned}\big\|Err^{\Psi}_{M,l}\big\|^2_{H^{1/2}} &\lesssim\big(1+|\log(v)|^2\big)\big(\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}\big)+\mathcal{M}_{l-1}\\ &\lesssim\big(1+|\log(v)|^2\big)\big(\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}\big)+\big\|Err^{\Psi}_{M,l-1}\big\|^2_{H^{1/2}}.\end{aligned}$$

By induction, we obtain the conclusion, since:

$$\big\|Err^{\Psi}_{M,l}\big\|^2_{H^{1/2}} \lesssim \big(1+|\log(v)|^2\big)\big(\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}\big)+\big\|Err^{\Psi}_{M,0}\big\|^2_{H^{1/2}} \lesssim \big(1+|\log(v)|^2\big)\big(\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}\big)$$

□

Proposition 9.5. *For any* $0 \leq l \leq \frac{n-4}{2}$ *and* $0 \leq m \leq M+\frac{n-4}{2}-l$*, the error terms* $Err^{\Psi}_{m,l}$ *satisfy the estimate:*

$$\big\|Err^{\Psi}_{m,l}\big\|^2_{L^2} \lesssim (1+|\log(v)|^2)\big(\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}\big). \tag{9.23}$$

Proof. As in the proof of Proposition 9.4, we use (9.17)-(9.18), the Bianchi equations in Proposition 2.6, and the null structure equations in Proposition 2.4:

$$\big\|v\mathcal{F}_{(m)(l+1)(l+m+1)}(\Psi)\big\|^2_{L^2} \lesssim (1+|\log(v)|^2)\sum_{i\leq l+1}\sum_{k\leq m}\big\|v\nabla^k\nabla_4^i\Psi\big\|^2_{L^2}+\big\|v\nabla^k\nabla_4^i\psi^*\big\|^2_{L^2} \lesssim$$

$$\lesssim (1+|\log(v)|^2)\bigg(\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}+\big\|v\nabla_4^{\frac{n-2}{2}}\psi^*\big\|^2_{H^M}+\big\|v\nabla_4^{\frac{n-2}{2}}\Psi\big\|^2_{H^M}\bigg) \lesssim$$

$$\lesssim (1+|\log(v)|^2)\Big(\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}+\big\|v\nabla_4^{\frac{n-4}{2}}\big(\Psi+\psi\psi^*\big)\big\|^2_{H^M}\Big)$$

$$+(1+|\log(v)|^2)\Big(\big\|\nabla_4^{\frac{n-4}{2}}\big(\nabla\Psi+\psi\Psi\big)\big\|^2_{L^2}+\big\|v\nabla_4\nabla^M\nabla_4^{\frac{n-4}{2}}\Psi\big\|^2_{L^2}\Big)$$

$$\lesssim (1+|\log(v)|^2)\big(\mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}\big).$$

Since $l \leq \frac{n-4}{2}$, we use (9.18) to get the bound:

$$\big\|v\mathcal{F}_{(m+1)(l)(l+m+1)}(\Psi)\big\|^2_{L^2} \lesssim \sum_{i\leq l}\sum_{k\leq m+1}\big\|v\nabla^k\nabla_4^i\Psi\big\|^2_{L^2}+\big\|v\nabla^k\nabla_4^i\psi^*\big\|^2_{L^2} \lesssim \mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}.$$

Similarly, we have that $l-1 \leq \frac{n-6}{2}$, so we only need to use the lower order curvature norm $\mathcal{L}$ and the Ricci coefficients norm $\mathcal{R}$ to bound:

$$\big\|\mathcal{F}_{(m+2)(l-1)(l+m+1)}(\Psi)\big\|^2_{L^2} \lesssim \sum_{i\leq l-1}\sum_{k\leq m+2}\big\|\nabla^k\nabla_4^i\Psi\big\|^2_{L^2}+\big\|\nabla^k\nabla_4^i\psi^*\big\|^2_{L^2} \lesssim \mathcal{L}+\mathcal{R}.$$

Analogously to (9.21), we use (9.18) to obtain:

$$\big\|\mathcal{F}_{(m)(l)(l+m)}(\Psi)\big\|^2_{L^2} \lesssim \sum_{i\leq l}\sum_{k\leq m}\big\|\nabla^k\nabla_4^i\Psi\big\|^2_{L^2}+\big\|\nabla^k\nabla_4^i\psi^*\big\|^2_{L^2} \lesssim \mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}. \tag{9.24}$$

As in the proof of Proposition 9.4, we recall that we have a similar result to Lemma 9.1 for the error terms $\mathcal{F}^{lot}$, which implies:

$$\begin{aligned}\big\|\mathcal{F}^{lot}_{(m+1)(l)(l+m+1)}(\Psi)\big\|^2_{L^2} &\lesssim \sum_{\substack{i\leq l\\ k\leq m}} \big\|\nabla^k\nabla_4^i\Psi\big\|^2_{L^2} + \sum_{\substack{i\leq l-1\\ k\leq m+1}} \big\|\nabla^k\nabla_4^i\Psi\big\|^2_{L^2} + \sum_{\substack{i\leq l\\ k\leq m+1}} \big\|\nabla^k\nabla_4^i\psi^*\big\|^2_{L^2}\\ &\lesssim \mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}.\end{aligned}$$

As in the proof of (9.22), we use the definitions of the curvature norms (9.1)-(9.3) to get the bound:

$$\big\|\nabla^m\nabla_4^l\big(\Psi\cdot\Psi^G\big)\big\|^2_{L^2} \lesssim \big(1+|\log(v)|^2\big)\big(\mathcal{T}+\mathcal{S}+\mathcal{L}\big). \tag{9.25}$$

We use (9.24), (9.25), and the formula $Ri\slashed{e}m = \Psi+\psi\psi$ (implied by the constraint equation (2.19)) to get that:

$$\sum_{i+2j=m} \big\|\nabla^i\big(Ri\slashed{e}m^{j+1}\cdot\nabla_4^l\Psi\big)\big\|^2_{L^2} \lesssim \mathcal{T}+\mathcal{S}+\mathcal{L}+\mathcal{R}.$$

Finally, to conclude the proof of (9.23), we use the commutation formulas in Lemma 2.5:

$$\big\|\nabla^m\nabla_4^l\nabla\big(\psi\psi\psi\big)\big\|^2_{L^2} \lesssim \mathcal{R}.$$

□

9.4 The Proof of Theorem 9.1

We combine the estimates established in Sections 9.2 and 9.3 to complete the proof of Theorem 9.1:

Proof of Theorem 9.1. We use the estimates (9.11), (9.12), (9.13), (9.15) for the norms $\mathcal{T}$, $\mathcal{S}$, $\mathcal{L}$, and $\mathcal{R}$, and the estimates (9.19), (9.23) for the error terms. Adding these together, we obtain that for all $v\leq 1$:

$$\mathcal{T}+\mathcal{L}+\mathcal{R}+\frac{1}{1+|\log v|^2}\mathcal{S} \lesssim \mathcal{D}+\epsilon\mathcal{R}+\int_0^v \big(1+|\log v'|^2\big)^2 v'^{-\frac{1}{2}}\left(\mathcal{T}+\frac{1}{1+|\log v'|^2}\mathcal{S}+\mathcal{L}+\mathcal{R}\right)dv'. \tag{9.26}$$

We first take $\epsilon>0$ small enough to absorb the $\mathcal{R}$ term, then apply Gronwall's inequality:

$$\mathcal{T}+\mathcal{L}+\mathcal{R}+\frac{1}{1+|\log v|^2}\mathcal{S} \lesssim \mathcal{D}. \tag{9.27}$$

By self-similarity, (9.27) implies the desired estimates for the Ricci coefficients and curvature components in the statement of Theorem 9.1. To complete the proof of the theorem, we also show that $\big\|\slashed{g}^*\big\|^2_{H^{M+1}} \lesssim \mathcal{D}$. We first need to prove this on the sphere $S_{-1,0}$. Similarly to the derivation of (9.9), we use (3.48) to get:

$$\sum_{m\leq M+1} \big\|\mathcal{L}_\theta^m \slashed{g}_0^*\big\|^2_{L^2(S^n)} \lesssim \big\|\slashed{g}_0^*\big\|^2_{\mathring{H}^{M+1}} \lesssim \mathcal{D}.$$

We then use (3.48) for covariant derivatives with respect to $\slashed{g}_0$ and prove by induction on $m \leq M+1$ that:

$$\big\|\nabla^m \slashed{g}_0^*\big\|^2_{L^2} \lesssim \mathcal{D}.$$

The conclusion follows by using the metric equation $\mathcal{L}_v \slashed{g}^* = \psi^*$ and the estimate $\mathcal{R} \lesssim \mathcal{D}$. $\square$

10 Estimates from $\{v = -u\}$ to $\{v = 0\}$

In this section we consider the smooth straight self-similar vacuum spacetime $(\mathcal{M}, g)$ obtained from small initial data on the sphere $S_{(-1,1)}$ and we prove optimal estimates on the induced asymptotic data set $\Sigma(\slashed{g}_0, h)$ at $\{u = -1,\ v = 0\}$. Using the ambient metric formulation, in the original $(n+1)$-dimensional formulation these correspond to proving estimates on the asymptotic data at $\mathcal{I}^-$ in terms of initial data at a finite time. This section is based on [Cic24, Section 10].

Theorem 10.1. *For any $M > 0$ large enough and $\epsilon > 0$ small enough we consider the smooth straight initial data on the sphere $S_{(-1,1)}$, with the initial data norm:*

$$\Xi_M^2 = \sum_{i+j=0}^{\frac{n-4}{2}} \sum_{m=0}^{M+\frac{n-4}{2}-i-j} \big\|\nabla_3^i \nabla_4^j \Psi\big\|^2_{H^{m+1}} + \sum_{i+j=0}^{\frac{n-4}{2}} \big\|\nabla^M \nabla_3^i \nabla_4^j \Psi\big\|^2_{H^{3/2}} + \sum_{i+j=0}^{\frac{n-2}{2}} \big\|\nabla^M \nabla_3^i \nabla_4^j \Psi\big\|^2_{H^{1/2}}$$
$$+ \sum_{k=0}^{1} \sum_{i+j=\frac{n-4}{2}-k} \big\|\nabla^{M+1+k} \nabla_3^i \nabla_4^j \psi^*\big\|^2_{H^{1/2}} + \sum_{i+j=0}^{\frac{n-4}{2}} \sum_{m=0}^{M+\frac{n-4}{2}-i-j} \big\|\nabla_3^i \nabla_4^j \psi^*\big\|^2_{H^{m+1}} + \big\|\slashed{g}^*\big\|^2_{H^{M+1}}.$$

We assume that the initial data satisfy the smallness assumption $\Xi_M \lesssim \epsilon$. We denote by $(\mathcal{M}, g)$ the smooth vacuum spacetime obtained using Theorems 3.1, 3.2, and 4.1, with induced asymptotic data $(\slashed{g}_0, h)$ at $S_{(-1,0)}$. The corresponding asymptotic data set $\Sigma(\slashed{g}_0, h)$ satisfies the estimate:

$$\big\|\Sigma(\slashed{g}_0, h)\big\|_M \lesssim \Xi_M, \tag{10.1}$$

where the asymptotic data norm of order $\big\|\Sigma\big\|^2_M$ is given in Definition 9.1.

We follow the strategy outlined in Section 1.3.3 of the introduction. Using self-similarity as in Section 9, it suffices to work on the null hypersurface $\{u = -1\}$. The idea of the proof is to define suitable norms in Section 10.1 for the curvature components and Ricci coefficients, which we then estimate one at a time in Section 10.2. Finally, we combine the estimates to complete the proof of Theorem 10.1 in Section 10.3.

Similarly to Section 9, we outline the steps of the proof in detail to assist the reader throughout the section. We define the top order curvature norm $\mathcal{T}$ in (10.2), which we estimate in Corollary 10.1 as a consequence of Theorem 8.1. We define the lower order curvature norms $\mathcal{L}$ in (10.3) and $\mathcal{M}_l$ in (10.4), which we estimate in Propositions 10.1 and 10.2. We define the Ricci coefficients norm $\mathcal{R}$ in (10.5) and estimate it in Proposition 10.3. As in Section 9, these estimates introduce nonlinear error terms, which we estimate in Propositions 10.4 and 10.5 in Section 10.2. Once again, in controlling the error terms we also use the estimate (10.9) for the lower order pointwise norms $\mathcal{P}$ in (10.6) and $\mathcal{SP}$ in (10.7), which follows from Section 3. In Section 10.3 we combine all these estimates to complete the proof of Theorem 10.1.

10.1 Norms

Following the above outline, we define the following norms on $\big\{u=-1,\ 0\leq v\leq 1\big\}$:

- Top order energy $\mathcal{T}=\mathcal{T}(-1,v)$:

$$\begin{aligned}\mathcal{T} =&v^2\big\|\nabla_4\nabla^M\nabla_4^{\frac{n-4}{2}}\alpha\big\|^2_{H^{1/2}}+\sum_{m=0}^{M}v\big\|\nabla^m\nabla_4^{\frac{n-4}{2}}\alpha\big\|^2_{H^{3/2}}+\sum_{m=0}^{M}\sqrt{v}\big\|\nabla^m\nabla_4^{\frac{n-4}{2}}\alpha\big\|^2_{H^{1/2}}+ \\ &+\sum_{m=0}^{M}\big\|\nabla^m\nabla_4^{\frac{n-4}{2}}\Psi^G\big\|^2_{H^{3/2}}+\sum_{m=0}^{M}v\big\|\nabla_4\nabla^m\nabla_4^{\frac{n-4}{2}}\Psi^G\big\|^2_{H^{1/2}}.\end{aligned} \tag{10.2}$$

- Lower order energy $\mathcal{L}=\mathcal{L}(-1,v)$:

$$\mathcal{L}=\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}v\big\|\nabla_4^{l+1}\Psi\big\|^2_{H^m}+\big\|\nabla_4^l\Psi\big\|^2_{H^{m+1}}. \tag{10.3}$$

- Fractional lower order energy $\mathcal{M}_l=\mathcal{M}_l(-1,v)$ for any $0\leq l\leq\frac{n-6}{2}$:

$$\mathcal{M}_l=\big\|\nabla^M\nabla_4^l\Psi\big\|^2_{H^{5/2}}. \tag{10.4}$$

- Ricci coefficients norm $\mathcal{R}=\mathcal{R}(-1,v)$:

$$\mathcal{R}=\sum_{k=0}^{1}\big\|\nabla^{M+1+k}\nabla_4^{\frac{n-4}{2}-k}\psi^*\big\|^2_{H^{1/2}}+\sum_{l=0}^{\frac{n-4}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\big\|\nabla_4^l\psi^*\big\|^2_{H^{m+1}}. \tag{10.5}$$

- Lower order pointwise norm $\mathcal{P} = \mathcal{P}(-1, v)$ for $N' = \frac{M}{2} + \frac{n}{4}$:

$$\mathcal{P} = \sum_{l=0}^{\frac{n-6}{2}} \sum_{m=0}^{N'} \big\|\nabla^m \nabla_4^l \Psi\big\|^2_{L^\infty} + \sum_{l=0}^{\frac{n-4}{2}} \sum_{m=0}^{N'} \big\|\nabla^m \nabla_4^l \psi^*\big\|^2_{L^\infty}. \tag{10.6}$$

- Mildly singular pointwise norm $\mathcal{SP} = \mathcal{SP}(-1, v)$:

$$\mathcal{SP} = \sum_{m=0}^{N'} \big\|\nabla^m \nabla_4^{\frac{n-4}{2}} \Psi\big\|^2_{L^\infty} + \sum_{m=0}^{N'} \big\|\nabla^m \nabla_4^{\frac{n-2}{2}} \psi^*\big\|^2_{L^\infty}. \tag{10.7}$$

- Initial data norm $\mathcal{D}$ at $(u, v) = (-1, 1)$:

$$\begin{aligned}\mathcal{D} :=& \sum_{l=0}^{\frac{n-4}{2}} \sum_{m=0}^{M+\frac{n-4}{2}-l} \big\|\nabla_4^l \Psi\big\|^2_{H^{m+1}} + \sum_{l=0}^{\frac{n-4}{2}} \big\|\nabla^M \nabla_4^l \Psi\big\|^2_{H^{3/2}} + \sum_{l=0}^{\frac{n-2}{2}} \big\|\nabla^M \nabla_4^l \Psi\big\|^2_{H^{1/2}} + \\ &+ \sum_{k=0}^{1} \sum_{l=\frac{n-4}{2}-k} \big\|\nabla^{M+1+k} \nabla_4^l \psi^*\big\|^2_{H^{1/2}} + \sum_{l=0}^{\frac{n-4}{2}} \sum_{m=0}^{M+\frac{n-4}{2}-l} \big\|\nabla_4^l \psi^*\big\|^2_{H^{m+1}} + \big\|\slashed{g}^*\big\|^2_{H^{M+1}}.\end{aligned} \tag{10.8}$$

We remark that using self-similarity, we can replace ∇_3 derivatives with ∇_4 derivatives. Thus, we obtain that $\Xi_M^2 \sim \mathcal{D}$, so it suffices to consider $\mathcal{D}$ as the initial data norm.

10.2 Estimates for the Norms

We start with some preliminary estimates for the pointwise norms $\mathcal{P}$ and $\mathcal{SP}$ which follow from the estimates of Section 3. Using Theorems 3.1, 4.1, and 3.2, we get that for $N' = \frac{M}{2} + \frac{n}{4}$ we have the estimates:

$$\mathcal{P} \leq \epsilon, \ \mathcal{SP} \leq \epsilon\big(1 + |\log(v)|^2\big). \tag{10.9}$$

The bound for the top order energy $\mathcal{T}$ follows from the refined estimates in Theorem 8.1. We can then bound the remaining norms using standard estimates. We note that as in Section 9, the nonlinear error terms Err^Ψ do not create significant difficulties.

As a consequence of Theorem 8.1, we obtain the following estimate for the top order energy (10.2):

Corollary 10.1. *The top order energy $\mathcal{T}$ satisfies the estimate for* $0 \leq v \leq 1$:

$$\mathcal{T} \lesssim \mathcal{D} + \sum_{m=0}^{M} \int_v^1 \big\|Err^\Psi_{m,\frac{n-4}{2}}\big\|^2_{H^{1/2}} dv'. \tag{10.10}$$

Proof. We recall that according to Section 5 and Remark 5.4:

$$\Phi_0 = \nabla_4^{\frac{n-4}{2}} \alpha, \ \Phi_i = \nabla_4^{\frac{n-4}{2}} \Psi^G, \ F^0_m = Err^\Psi_{m,\frac{n-4}{2}}, \ F^i_m = Err^\Psi_{m,\frac{n-4}{2}}$$

satisfy the second model system as defined in (5.14) and also [Cic26, Definition 1.1]. The bounds on the background $\big(\mathcal{M}, g\big)$ required in [Cic26, Theorem 1.2] follow by Theorem 3.1. Moreover, since $\mathcal{D} \lesssim \epsilon^2$, we have in particular that $\|\psi\|_{H^{M+1}}(-1,1) \lesssim 1$, so the implicit constant in Theorem 8.1 depends only on M. Thus, we apply Theorem 8.1 to obtain the following estimate:

$$\sum_{m=0}^{M} \sqrt{v}\big\|\nabla^m\nabla_4^{\frac{n-4}{2}}\alpha\big\|^2_{H^{1/2}} + \sum_{m=0}^{M} v\big\|\nabla^m\nabla_4^{\frac{n-4}{2}}\alpha\big\|^2_{H^{3/2}} + v^2\big\|\nabla_4\nabla^M\nabla_4^{\frac{n-4}{2}}\alpha\big\|^2_{H^{1/2}}$$

$$+\int_v^1 \big\|\nabla_4^{\frac{n-4}{2}}\alpha\big\|^2_{H^{M+1}}dv' + \sum_{m=0}^{M} \big\|\nabla^m\nabla_4^{\frac{n-4}{2}}\Psi^G\big\|^2_{H^{3/2}}$$

$$+\sum_{m=0}^{M} v\big\|\nabla_4\nabla^m\nabla_4^{\frac{n-4}{2}}\Psi^G\big\|^2_{H^{1/2}} + \sum_{m=0}^{M}\int_v^1 \big\|\nabla_4\nabla^m\nabla_4^{\frac{n-4}{2}}\Psi^G\big\|^2_{H^{1/2}}dv' \lesssim$$

$$\lesssim \sum_{m=0}^{M}\Big(\big\|\nabla^m\nabla_4^{\frac{n-4}{2}}\Psi\big\|^2_{H^{3/2}} + \big\|\nabla^m\nabla_4^{\frac{n-2}{2}}\Psi\big\|^2_{H^{1/2}}\Big)\Big|_{v=1} + \sum_{m=0}^{M}\int_v^1 \big\|Err^{\Psi}_{m,\frac{n-4}{2}}\big\|^2_{H^{1/2}}dv'.$$

We bound the initial data term using $\mathcal{D}$ to obtain the conclusion. □

We prove the following result for the lower order energy $\mathcal{L}$ defined in (10.3):

Proposition 10.1. *The lower order energy $\mathcal{L}$ satisfies the estimate for $0 \leq v \leq 1$:*

$$\mathcal{L} \lesssim \mathcal{D} + \epsilon\mathcal{R} + \int_v^1 (v')^{-1/2}\mathcal{T}dv' + \sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_v^1 \Big\|Err^{\Psi}_{m,l}\Big\|^2_{L^2}dv'. \tag{10.11}$$

Proof. As in the proof of Proposition 9.1 in Section 9, we contract (5.11) with $2\nabla_4\nabla^m\nabla_4^l\Psi$ and sum for all $0 \leq l \leq \frac{n-6}{2}$, $0 \leq m \leq M + \frac{n-4}{2} - l$, and every curvature component Ψ. We obtain a good bulk term since $n - 2l - 5 \geq 1$:

$$\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l} v\big\|\nabla_4\nabla^m\nabla_4^l\Psi\big\|^2_{L^2} + \big\|\nabla\nabla_4^l\Psi\big\|^2_{H^m} + \int_v^1 \big\|\nabla_4\nabla^m\nabla_4^l\Psi\big\|^2_{L^2}dv'$$

$$\lesssim \mathcal{D} + \sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_v^1 \big\|[\nabla,\nabla_4]\nabla^m\nabla_4^l\Psi\big\|^2_{L^2} + \big\|\nabla^{m+1}\nabla_4^l\Psi\big\|^2_{L^2}dv' + \sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_v^1 \big\|Err^{\Psi}_{m,l}\big\|^2_{L^2}.$$

We also have the estimate:

$$\big\|\nabla_4^l\Psi\big\|^2_{L^2} \lesssim \mathcal{D} + \int_v^1 \big\|\nabla_4^{l+1}\Psi\big\|^2_{L^2} \lesssim \ldots \lesssim \mathcal{D} + \int_v^1 \big\|\nabla_4^{\frac{n-4}{2}}\Psi\big\|^2_{L^2} \lesssim \mathcal{D} + \int_v^1 (v')^{-1/2}\mathcal{T}dv'.$$

We use the commutation formulas in Lemma 2.7 and Gronwall to obtain:

$$\sum\nolimits_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l} v\big\|\nabla_4\nabla^m\nabla_4^l\Psi\big\|^2_{L^2} + \big\|\nabla_4^l\Psi\big\|^2_{H^{m+1}} \tag{10.12}$$

$$\lesssim \mathcal{D}+\int_v^1 (v')^{-1/2}\mathcal{T}dv'+\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_v^1 \left\|Err^{\Psi}_{m,l}\right\|^2_{L^2}.$$

Next, we use the commutation formulas in Lemma 2.7 to obtain:

$$\mathcal{L}\lesssim \sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l} v\left\|\psi\nabla_4^l\Psi\right\|^2_{H^m}+\mathcal{D}+\int_v^1 (v')^{-1/2}\mathcal{T}dv'+\sum_{l=0}^{\frac{n-6}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_v^1 \left\|Err^{\Psi}_{m,l}\right\|^2_{L^2}dv'. \quad (10.13)$$

Similarly to the proof of Proposition 9.1, we bound the first term in (10.13) by using the lower order pointwise bound (10.9) and the bound in (10.12) for $\left\|\nabla_4^l\Psi\right\|^2_{H^{m+1}}$ in order to conclude (10.11). ☐

The bounds for the remaining terms are similar to Section 9. We notice that the norms defined in Section 10.1 control the same terms as the norms of Section 9.1, the only difference being the bound:

$$\sum_{m=0}^{M}\left\|\nabla^m\nabla_4^{\frac{n-4}{2}}\alpha\right\|^2_{H^{1/2}}\lesssim v^{-1/2}\mathcal{T}. \quad (10.14)$$

Due to the structure of the error terms it suffices to control only $M+1/2$ angular derivatives of $\nabla_4^{\frac{n-4}{2}}\alpha$ in (10.14), since the terms with more angular derivatives also have better v weights. Thus, the bound (10.14) replaces the use of the mildly singular top order energy (9.2) in Section 9. We briefly explain the proofs for the remaining terms:

Proposition 10.2. *The fractional lower order energy* $\mathcal{M}_l$ *satisfies the estimate for any* $0\leq l\leq \frac{n-6}{2}$ *and* $0\leq v\leq 1:$

$$\mathcal{M}_l\lesssim v^{-1/2}\mathcal{T}+\mathcal{L}+\mathcal{R}+\left\|Err^{\Psi}_{M,l}\right\|^2_{H^{1/2}}. \quad (10.15)$$

Proof. As in the proof of Proposition 9.2 in Section 9, we have the estimate for any $0\leq l\leq\frac{n-6}{2}$:

$$\left\|\nabla^M\nabla_4^l\Psi\right\|^2_{H^{5/2}}\lesssim\mathcal{L}+\left\|v\nabla_4\nabla^M\nabla_4^{l+1}\Psi\right\|^2_{H^{1/2}}+\left\|v\nabla_4\nabla^M\left(\psi\nabla_4^l\Psi\right)\right\|^2_{H^{1/2}}+$$

$$+\left\|\nabla^M\nabla_4^{l+1}\Psi\right\|^2_{H^{1/2}}+\left\|\nabla^M\left(\psi\nabla_4^l\Psi\right)\right\|^2_{H^{1/2}}+\left\|Err^{\Psi}_{M,l}\right\|^2_{H^{1/2}}$$

$$\lesssim\mathcal{L}+v^{-1/2}\mathcal{T}+\mathcal{R}+\left\|v\nabla_4\nabla^M\nabla_4^{l+1}\Psi\right\|^2_{H^{1/2}}+\left\|v\nabla_4\nabla^M\left(\psi\nabla_4^l\Psi\right)\right\|^2_{H^{1/2}}+\left\|Err^{\Psi}_{M,l}\right\|^2_{H^{1/2}}.$$

Considering separately the cases $l=\frac{n-6}{2}$ and $0\leq l\leq\frac{n-8}{2}$, the same proof as in Proposition 9.2 implies:

$$\left\|v\nabla_4\nabla^M\nabla_4^{l+1}\Psi\right\|^2_{H^{1/2}}\lesssim v^{-1/2}\mathcal{T}+\mathcal{L}+\mathcal{R}.$$

Finally, we conclude since we have the bound similarly to the proof of Proposition 9.2:

$$\big\|v\nabla_4\nabla^M\big(\psi\nabla_4^l\Psi\big)\big\|^2_{H^{1/2}}\lesssim\big\|v\nabla^M\big(\psi\nabla_4^{l+1}\Psi\big)\big\|^2_{H^{1/2}}+\big\|v\nabla^M\big((\Psi+\psi\psi)\nabla_4^l\Psi\big)\big\|^2_{H^{1/2}}+\mathcal{L}+\mathcal{R}$$

$$\lesssim v^{-1/2}\mathcal{T}+\mathcal{L}+\mathcal{R}.$$

□

Next, we follow the same steps as in Section 9 to prove the result for the Ricci coefficients $\mathcal{R}$ defined in (10.5):

Proposition 10.3. *The Ricci coefficients norm $\mathcal{R}$ satisfies the estimate for any* $0\leq v\leq 1:$

$$\mathcal{R}\lesssim\mathcal{D}+\int_v^1\big((v')^{-1/2}\mathcal{T}+\mathcal{L}+\mathcal{R}+(v')^{1/2}\mathcal{M}_{\frac{n-6}{2}}\big)dv'. \tag{10.16}$$

Proof. Using the commuted equations (9.16) for the Ricci coefficients ψ^*, we obtain the estimate:

$$\sum_{l=0}^{\frac{n-4}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\big\|\nabla_4^l\psi^*\big\|^2_{H^{m+1}}\lesssim\mathcal{D}+\sum_{l=0}^{\frac{n-4}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_v^1(v')^{1/2}\big\|\nabla_4^l\big(\psi\psi^*\big)\big\|^2_{H^{m+1}}dv'$$

$$+\sum_{l=0}^{\frac{n-4}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_v^1(v')^{1/2}\big\|\psi\nabla_4^l\big(\psi\psi^*\big)\big\|^2_{H^{m+1}}dv'+\sum_{l=0}^{\frac{n-4}{2}}\sum_{m=0}^{M+\frac{n-4}{2}-l}\int_v^1(v')^{1/2}\big\|\nabla_4^l\Psi\big\|^2_{H^{m+1}}dv'$$

$$\lesssim\mathcal{D}+\int_v^1\big((v')^{-1/2}\mathcal{T}+\mathcal{L}+\mathcal{R}\big)dv'.$$

As in the proof of Proposition 9.3, we use the LP projections in Section 6 and Gronwall to obtain the following fractional estimate:

$$\big\|\nabla^{M+1}\nabla_4^{\frac{n-4}{2}}\psi^*\big\|^2_{H^{1/2}}\lesssim\mathcal{D}+\int_v^1\big((v')^{-1/2}\mathcal{T}+\mathcal{L}+\mathcal{R}\big)dv'+\int_0^v(v')^{1/2}\big\|\nabla_4\nabla^{M+1}\nabla_4^{\frac{n-4}{2}}\psi^*\big\|^2_{H^{1/2}}$$

$$\lesssim\mathcal{D}+\int_v^1\big((v')^{-1/2}\mathcal{T}+\mathcal{L}+\mathcal{R}\big)dv'+\sum_{m=0}^{M+1}\int_v^1\big\|\nabla^m\nabla_4^{\frac{n-4}{2}}\big(\psi\psi^*\big)\big\|^2_{H^{1/2}}+(v')^{1/2}\big\|\nabla^m\nabla_4^{\frac{n-4}{2}}\Psi\big\|^2_{H^{1/2}}$$

$$\lesssim\mathcal{D}+\int_v^1\big((v')^{-1/2}\mathcal{T}+\mathcal{L}+\mathcal{R}\big)dv'.$$

Similarly to Proposition 9.3, we also get the estimate in the case when $n\geq 6$:

$$\big\|\nabla^{M+2}\nabla_4^{\frac{n-6}{2}}\psi^*\big\|^2_{H^{1/2}}\lesssim\mathcal{D}+\int_v^1\big((v')^{-1/2}\mathcal{T}+\mathcal{L}+\mathcal{R}\big)dv'+\int_0^v(v')^{1/2}\big\|\nabla_4\nabla^{M+2}\nabla_4^{\frac{n-6}{2}}\psi^*\big\|^2_{H^{1/2}}$$

$$\lesssim\mathcal{D}+\int_v^1\big((v')^{-1/2}\mathcal{T}+\mathcal{L}+\mathcal{R}\big)dv'+\sum_{m=0}^{M+2}\int_0^v\big\|\nabla^m\nabla_4^{\frac{n-6}{2}}\big(\psi\psi^*\big)\big\|^2_{H^{1/2}}+(v')^{1/2}\big\|\nabla^m\nabla_4^{\frac{n-6}{2}}\Psi\big\|^2_{H^{1/2}}$$

$$\lesssim \mathcal{D} + \int_v^1 (v')^{-1/2}\mathcal{T} + \mathcal{L} + \mathcal{R} + (v')^{1/2}\mathcal{M}_{\frac{n-6}{2}}.$$

$\square$

For the rest of the section, we explain how to control the error terms from the above estimates. Since the lower order pointwise norms $\mathcal{P}$ and $\mathcal{SP}$ satisfy the same bounds as in Section 9 (see (9.10) and (10.9)), we get that Lemma 9.1 applies in the current situation. We adapt the proof of Proposition 9.4 to prove the following result for the error terms:

Proposition 10.4. *For any* $0 \leq l \leq \frac{n-4}{2}$, $0 \leq m \leq M$, *the error terms* $Err^{\Psi}_{m,l}$ *satisfy the estimate:*

$$\left\|Err^{\Psi}_{m,l}\right\|^2_{H^{1/2}} \lesssim \left(1 + |\log(v)|^2\right)\left(v^{-1/2}\mathcal{T} + \mathcal{L} + \mathcal{R}\right). \tag{10.17}$$

Proof. As in the proof of Proposition 9.4 in Section 9, the proof of (10.17) follows from the fractional lower order energy estimate (10.15), once we prove the claim:

$$\left\|Err^{\Psi}_{m,l}\right\|^2_{H^{1/2}} \lesssim \left(1 + |\log(v)|^2\right)\left(v^{-1/2}\mathcal{T} + \mathcal{L} + \mathcal{R}\right) + \mathcal{M}_{l-1}. \tag{10.18}$$

However, the proof of this estimate follows the exact same steps as in the proof of Proposition 9.4 once we replace all the mildly singular norms $\mathcal{S}$ by $v^{-1/2}\mathcal{T}$ using (10.14), so we omit repeating the proof here. We then use the fractional lower order energy estimate (9.14) to obtain:

$$\left\|Err^{\Psi}_{m,l}\right\|^2_{H^{1/2}} \lesssim \left(1 + |\log(v)|^2\right)\left(v^{-1/2}\mathcal{T} + \mathcal{L} + \mathcal{R}\right) + \mathcal{M}_{l-1}$$

$$\lesssim \left(1 + |\log(v)|^2\right)\left(v^{-1/2}\mathcal{T} + \mathcal{L} + \mathcal{R}\right) + \left\|Err^{\Psi}_{m,l-1}\right\|^2_{H^{1/2}}.$$

By induction, we obtain that:

$$\left\|Err^{\Psi}_{m,l}\right\|^2_{H^{1/2}} \lesssim \left(1 + |\log(v)|^2\right)\left(v^{-1/2}\mathcal{T} + \mathcal{L} + \mathcal{R}\right) + \left\|Err^{\Psi}_{m,0}\right\|^2_{H^{1/2}}$$

$$\lesssim \left(1 + |\log(v)|^2\right)\left(v^{-1/2}\mathcal{T} + \mathcal{L} + \mathcal{R}\right),$$

completing the proof of (10.17). $\square$

Similarly, we can repeat the proof of Proposition 9.5 and replace the mildly singular norms $\mathcal{S}$ by $v^{-1/2}\mathcal{T}$ using (10.14) in order to establish the following result for the remaining error terms:

Proposition 10.5. *For any for any* $0 \leq l \leq \frac{n-4}{2}$ *and* $0 \leq m \leq M + \frac{n-4}{2} - l$, *the error terms* $Err^{\Psi}_{m,l}$ *satisfy the estimate:*

$$\left\|Err^{\Psi}_{m,l}\right\|^2_{L^2} \lesssim \left(1 + |\log(v)|^2\right)\left(v^{-1/2}\mathcal{T} + \mathcal{L} + \mathcal{R}\right). \tag{10.19}$$

Proof. The proof follows the same steps as Proposition 9.5, so we omit repeating it here. $\square$

10.3 The Proof of Theorem 10.1

We combine the estimates established in Section 10.2 to complete the proof of Theorem 10.1:

Proof of Theorem 10.1. We use the estimates (10.10), (10.11), (10.16) for the norms $\mathcal{T}$, $\mathcal{L}$, and $\mathcal{R}$, and the estimates (10.17), (10.19) for the error terms. Adding these together, we obtain that for all $0 \leq v \leq 1$:

$$\mathcal{T} + \mathcal{L} + \mathcal{R} \lesssim \mathcal{D} + \epsilon\mathcal{R} + \int_v^1 \left(1 + |\log v'|^2\right)\left((v')^{-1/2}\mathcal{T} + \mathcal{L} + \mathcal{R}\right)dv'. \tag{10.20}$$

Taking $\epsilon > 0$ small enough and using Gronwall's inequality, we show that:

$$\mathcal{T} + \mathcal{L} + \mathcal{R} \lesssim \mathcal{D}. \tag{10.21}$$

By self-similarity, (10.21) implies the desired estimates for the Ricci coefficients and curvature components in the statement of Theorem 10.1. In order to complete the proof of Theorem 10.1, we prove the bound:

$$\left\|\slashed{g}^*_0\right\|^2_{\mathring{H}^{M+1}} + \left\|\mathcal{O}\right\|^2_{H^{M+1}} + \left\|\mathfrak{h}\right\|^2_{H^{M+1}} \lesssim \mathcal{D}, \tag{10.22}$$

Using the estimates on the asymptotic data in Theorem 8.3 we obtain:

$$\left\|\mathcal{O}\right\|^2_{H^{M+1}} \lesssim \mathcal{D} + \sum_{m=0}^{M} \int_0^1 \left\|Err^{\Psi}_{m,\frac{n-4}{2}}\right\|^2_{H^{1/2}} dv.$$

We then use the estimates (10.17), (10.19) on the error terms and (10.21):

$$\left\|\mathcal{O}\right\|^2_{H^{M+1}} \lesssim \mathcal{D} + \int_0^1 \left(1 + |\log(v)|^2\right)\left(v^{-1/2}\mathcal{T} + \mathcal{L} + \mathcal{R}\right)dv \lesssim \mathcal{D}. \tag{10.23}$$

Using the metric equation $\mathcal{L}_v\slashed{g}^* = \psi^*$ and the estimate $\mathcal{R} \lesssim \mathcal{D}$, we obtain that $\left\|\slashed{g}^*\right\|^2_{H^{M+1}} \lesssim \mathcal{D}$. We use this, together with the estimates in Proposition 3.8 and (3.48) in order to prove by induction on $m \leq M+1$ that $\left\|\mathcal{L}^m_\theta \slashed{g}^*_0\right\|^2_{L^2(S^n)} \lesssim \epsilon$. In particular, this allows us to bound the Christoffel symbols terms in (3.48) and prove:

$$\sum_{m \leq M+1} \left\|\mathcal{L}^m_\theta \slashed{g}^*_0\right\|^2_{L^2(S^n)} \lesssim \mathcal{D}.$$

We can then use (3.48) for covariant derivatives with respect to $\slashed{g}_{S^n}$ and prove by induction on $m \leq M+1$ that:

$$\big\|\mathring{\nabla}^m \slashed{g}_0^*\big\|_{L^2}^2 \lesssim \mathcal{D}. \tag{10.24}$$

In order to complete the proof of (10.22), we must bound $\big\|\mathfrak{h}\big\|_{H^{M+1}}$. By Theorem 8.1 we get:

$$\big\|\mathfrak{h}\big\|_{H^{M+1}}^2 \lesssim \mathcal{D} + \big\|h\big\|_{L^2}^2 + C\Big(\big\|\slashed{Ric}m_0\big\|_{H^M}^2\Big)\big\|\mathcal{O}\big\|_{H^M}^2 + \sum_{m=0}^{M}\int_0^1 \Big\|Err^{\Psi}_{m,\frac{n-4}{2}}\Big\|_{H^{1/2}}^2 dv \lesssim \mathcal{D} + \big\|h\big\|_{L^2}^2,$$

where we used the constraint equation (2.19), the bound (10.23) for $\mathcal{O}$, and the estimates (10.17), (10.19) on the error terms. For the rest of the proof we prove a suitable lower order estimate for $\big\|h\big\|_{L^2}^2$. As in the proof of Theorem 8.1, we define $\xi = \nabla_4^{\frac{n-4}{2}}\alpha$ and $\overline{\xi} = \xi - v\log v\nabla_4\xi$. We compute using (5.9):

$$\nabla_4\overline{\xi} = -\log v\cdot\nabla_4\big(v\nabla_4\xi\big) = \log v\cdot\Big(\Delta\nabla_4^{\frac{n-4}{2}}\alpha + \psi\nabla\nabla_4^{\frac{n-4}{2}}\Psi + Err^{\Psi}_{0,\frac{n-4}{2}}\Big).$$

Therefore, we get using the estimates (10.10) and (10.19):

$$\big\|h\big\|_{L^2}^2 = \lim_{v\to 0}\big\|\overline{\xi}\big\|_{L^2}^2 \lesssim \mathcal{D} + \int_0^1 (\log v)^2\cdot\Big(\big\|\nabla_4^{\frac{n-4}{2}}\Psi\big\|_{H^2}^2 + \big\|Err^{\Psi}_{0,\frac{n-4}{2}}\big\|_{L^2}^2\Big)dv \lesssim \mathcal{D}.$$

This completes the proof of Theorem 10.1. □

11 The Scattering Map

In this section, we follow the outline in Section 1.3.3 of the introduction and we put together our previous results in order to complete the proof of the third statement in Theorem 1.3. This section is based on [Cic24, Section 11]. For any $M > 0$ large enough and $\epsilon > 0$ small enough we consider smooth straight initial data $\big(\slashed{g}_0, h\big)$ such that the corresponding asymptotic data set satisfies $\Sigma\big(\slashed{g}_0, h\big) \in B^M_\epsilon\big(\Sigma_{\text{Minkowski}}\big)$. By Theorem 3.1 there exists a unique smooth straight self-similar vacuum solution $(\mathcal{M}, g)$ in double null coordinates defined in $\{u < 0,\ v > 0\}$ with asymptotic initial data at $\{v = 0\}$ given by $\big(\slashed{g}_0, h\big)$. Moreover, this induces smooth asymptotic data $\big(\underline{\slashed{g}_0}, \underline{h}\big)$ at $\{u = 0\}$ by Theorem 4.1. The estimates in Theorem 9.1 imply that:

$$\Xi_M \lesssim \Big\|\Sigma\big(\slashed{g}_0, h\big)\Big\|_M.$$

We also have by Theorem 10.1 that the reverse inequality holds. We use this inequality for the spacetime obtained by reversing the time orientation, namely by replacing (u, v) with $(-v, -u)$.

We obtain the estimate:

$$\Big\|\Sigma\big(\underline{\not\phi_0},\underline{h}\big)\Big\|_M \lesssim \Xi_M,$$

where $\Sigma\big(\underline{\not\phi_0},\underline{h}\big)$ is defined according to Remark 9.1. Therefore, there exists a constant $C_M>0$ depending only on M such that:

$$\Big\|\Sigma\big(\underline{\not\phi_0},\underline{h}\big)\Big\|_M \leq C_M\Big\|\Sigma\big(\not\phi_0,h\big)\Big\|_M. \tag{11.1}$$

We obtain that $\Sigma\big(\underline{\not\phi_0},\underline{h}\big)\in B^M_{C_M\epsilon}\big(\Sigma_{\text{Minkowski}}\big)$, so we can define the scattering map:

$$\mathcal{S}:B^M_{\epsilon}\big(\Sigma_{\text{Minkowski}}\big)\to B^M_{C_M\epsilon}\big(\Sigma_{\text{Minkowski}}\big),\ \mathcal{S}\big(\Sigma(\not\phi_0,h)\big)=\Sigma(\underline{\not\phi_0},\underline{h}).$$

Using the uniqueness of scattering states statement from Theorem 3.1, we obtain that $\mathcal{S}$ is injective. Moreover, we can apply the existence and uniqueness results, together with the estimate (1.17) in the reverse time direction to obtain that $B^M_{\epsilon/C_M}\big(\Sigma_{\text{Minkowski}}\big)\subset\mathcal{S}\big(B^M_{\epsilon}(\Sigma_{\text{Minkowski}})\big)$. Therefore, $\mathcal{S}$ is locally invertible at $\Sigma_{\text{Minkowski}}$. Finally, the estimate (11.1) implies that $\mathcal{S}$ is locally Lipschitz at $\Sigma_{\text{Minkowski}}$.

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