Extension criterion for the spherically-symmetric Einstein–Maxwell-charged-scalar-field equations with absolutely continuous initial data

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Abstract

We prove an extension criterion at the center for the massless Einstein–Maxwell–charged-scalar-field system in spherical symmetry. For an AC solution with a regular center and an untrapped past region, we show that a bounded blue-shift factor along the terminal ingoing cone implies that the solution can be continued beyond the endpoint of the cone. We point out that no uniform blue-shift bound on the preceding ingoing cones is assumed.

As a particular case, our result also supplies an extension criterion for the real, uncharged Einstein–scalar-field system in spherical symmetry studied by Christodoulou. In this setting, our result supplies the continuation input in Christodoulou's framework for weak and strong cosmic censorship for spherically symmetric initial data in the absolutely continuous class considered.

For the Einstein–Maxwell–charged-scalar-field system, An–Chen–He recently established a related extension criterion for C^1 characteristic data under a uniform blue-shift bound throughout the past characteristic region, using an independent approach. Our strategy differs in that it is based on a new identity on time-like curves and a resulting observation estimate on these curves.

Formal verification and AI statement: The proof was obtained through interactions with AI models (ChatGPT Astra 6, Claude Opus 5.5) prompted by Serban Cicortas, Christoph Kehle, Yakov Shlapentokh-Rothman and Maxime Van de Moortel. Our result provides an answer to a question that has been of interest to many people in the general relativity community, and it builds on several results in the literature. At the present moment, the proof below is a working draft written using AI, but we will undertake a team effort to completely rewrite the proof in the near future, with an emphasis towards exposition. Finally, the proof below has been formally verified using Lean 4, see here.

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1 Introduction

1.1 Main result and uncharged cosmic-censorship consequences

Theorem 1.1 concerns spherically symmetric solutions (g, F, ϕ) of the massless Einstein–Maxwell–charged-scalar-field equations (1.1) below. We prove that, under the geometric and regularity assumptions stated in the theorem, an AC solution extends through a central endpoint if the blueshift factor is bounded on the last ingoing cone. We do not assume a uniform bound on the preceding ingoing cones.

In the real uncharged case, we combine this extension criterion with Christodoulou's instability and global-structure results to deduce weak and strong cosmic censorship in the AC class, in the precise sense of Corollary 1.3. For charged collapse, a further instability argument is needed to exclude the remaining central singularities generically.

1.1.1 Equations and conventions

Throughout, the cosmological constant and magnetic charge vanish.

We use the Lorentzian signature $(-,+,+,+)$ and define

$$D_\mu = \nabla_\mu + iq_0 A_\mu, \qquad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu,$$

where ϕ is complex-valued, A is a real electromagnetic potential, and $q_0 \in \mathbb{R}$ is the scalar charge. The massless Einstein–Maxwell–charged-scalar-field equations, with vanishing cosmological constant, are

$$\mathrm{Ric}(g)_{\mu\nu} = 2\,\mathrm{Re}\Big(D_\mu\phi\,\overline{D_\nu\phi}\Big) + T^{\mathrm{EM}}_{\mu\nu}, \tag{1.1a}$$

$$T^{\mathrm{EM}}_{\mu\nu} = 2\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right), \tag{1.1b}$$

$$D^\alpha D_\alpha \phi = 0, \tag{1.1c}$$

$$\nabla^\mu F_{\mu\nu} = q_0\,\mathrm{Im}\Big(\overline{\phi}\,D_\nu\phi\Big), \tag{1.1d}$$

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0. \tag{1.1e}$$

At the AC/BV regularity considered here, solving this system means satisfying its double-null reduction in the integral sense specified below. This normalization agrees with $\mathrm{Ric}(g) = 2\,d\phi\otimes d\phi$ in the real, uncharged case with $F = 0$. The regular center imposes zero central electric charge. Write

$$g = -\Omega^2\,\mathrm{d}u\,\mathrm{d}v + r^2\,\mathrm{d}\omega^2, \qquad \lambda = \partial_v r > 0, \qquad \nu = \partial_u r < 0, \qquad \mu = \frac{2m}{r}, \qquad 1-\mu = -\frac{4\nu\lambda}{\Omega^2} > 0.$$

Here m is the ordinary Hawking mass, and Q is the electric charge enclosed by a symmetry sphere, with sign convention

$$F = \frac{\Omega^2 Q}{2r^2}\,du\wedge dv.$$

We use q_0 for the fixed scalar charge. Our m is the Hawking mass denoted by ρ there. The gauge is $A_v = 0$, $A_u(u,u) = 0$, so $D_u = \partial_u + iq_0A_u$ and $D_v = \partial_v$. Set

$$\theta = rD_v\phi, \quad \xi = rD_u\phi, \quad \alpha = D_v(r\phi) = \lambda\phi + \theta, \quad \beta = D_u(r\phi) = \nu\phi + \xi, \quad \alpha' = \lambda^{-1}D_v(\alpha/\lambda).$$

1.1.2 Solution class and terminal-cone extension

The original development is

$$\mathcal{Q} = \{u_i \le u < 0, \quad u \le v \le 0\}, \qquad u_i < 0.$$

Its regular axis is $\Gamma = \{u = v < 0\}$. A proper compact subset avoids $(0,0)$ but may meet this existing axis and the last cone $v = 0$. We use the following precise local characteristic AC/BV class, on any domain $D \subset \{u \le v\}$ under consideration. We set $\mu = 0$ on the existing regular axis.

1. The nine primary fields $r, \lambda, \nu, m, Q, \phi, A_u, \theta, \xi$ are continuous on D, including its existing axis. The signs $\lambda > 0$, $\nu < 0$, and $1 - \mu > 0$ hold, and $r > 0$ off the axis. In particular, both expansions and their inverses are bounded on each compact subset of D.

2. The reduced equations hold in integral form on every closed characteristic segment in D that does not meet the axis. Passage to an axis endpoint is justified by the estimates below. The fields r, m, Q, ϕ are AC on every compact characteristic segment in D. The outgoing field α/λ is AC on every compact outgoing segment, including segments starting at an existing axis point.

3. For every compact $K \subset D$, one finite constant bounds the variations of $\lambda, \nu, \alpha/\lambda, \beta/\nu$ on all outgoing and all incoming segments contained in K. The variations of ϕ vanish uniformly on shrinking triangles about each existing axis point.

4. At the axis,

$$r = m = Q = 0, \qquad \lambda + \nu = 0, \qquad \alpha/\lambda = \beta/\nu = \phi, \qquad \mu \longrightarrow 0.$$

   Variations are ordinary variations in the specified gauge $A_v = 0$. When covariant incoming variation is used, it means variation after unitary parallel transport along that incoming section.

The traces and continuity requirements are local at existing axis points; none is imposed at $(0,0)$. These are proper-development hypotheses, not uniform terminal variation bounds. Section 3 constructs this class from AC characteristic data of small total variation. No Hölder or Dini modulus, pointwise derivative bound, or L^2 bound for α' is assumed.

In this initial gauge, absolute continuity means

$$\frac{\alpha}{\lambda}(r) = \frac{\alpha}{\lambda}(0) + \int_0^r \alpha'(s)\,\mathrm{d}s, \qquad \alpha' \in L^1([0,R]), \qquad \mathrm{TV}_{[0,R]}(\alpha/\lambda) = \int_0^R |\alpha'(r)|\,\mathrm{d}r.$$

In particular, the derivative integrals tend uniformly to zero on intervals whose lengths tend to zero. General BV data can also have a singular measure in their distributional derivative; the theorem is stated for the AC subclass.

Theorem 1.1 (Charged terminal-cone extension)**.** *Let (g, F, ϕ) be a spherically symmetric solution of (1.1) on $\mathcal{Q}$ in the local characteristic AC/BV class specified above. Assume $0 \le \mu < 1$ and*

$$r(u_i, v) = \frac{v - u_i}{2}, \qquad (\alpha/\lambda)(u_i, \cdot) \in \mathrm{AC}[u_i, 0].$$

Suppose the terminal blue-shift factor is bounded, equivalently that its logarithm satisfies:

$$\sup_{u_i < u < 0} \int_{u_i}^{u} \frac{(-\nu)(\mu - Q^2/r^2)}{(1-\mu)r}(s,0)\,\mathrm{d}s < \infty. \tag{1.2}$$

Then there are $u_i < u_* < 0 < v_+$ and a solution in the preceding class on

$$\mathcal{Q} \cup \{u_* \le u \le v \le v_+\},$$

in the same coordinates and gauge, equal to the original solution on $\mathcal{Q}$. Thus $(0,0)$ is an interior point of the extended axis, where $A_u = 0$. The continuation asserted is in the local characteristic AC/BV class. In particular, in the stated regular-axis electromagnetic gauge,

$$\mathrm{TV}_{[u_i,0)}\,\phi(\cdot,0) + \mathrm{TV}_{[u_i,0)}(\beta/\nu)(\cdot,0) + \mathrm{TV}_{[u_i,0)}\log(-\nu)(\cdot,0) < \infty. \tag{1.3}$$

The positive integrals on complete incoming cones are uniformly bounded:

$$\sup_{u_i<v<0}\int_{u_i}^{v}\frac{(-\nu)\mu}{(1-\mu)r}(u,v)\,\mathrm{d}u < \infty. \tag{1.4}$$

Under the standing solution-class and data assumptions alone, the terminal hypothesis is equivalent to finiteness of

$$\int_{u_i}^{0}\frac{(-\nu)\mu}{(1-\mu)r}(u,0)\,\mathrm{d}u. \tag{1.5}$$

The coupling q_0 and the size of the original datum need not be small.

On the last cone $(-\nu)\,\mathrm{d}u = |\,\mathrm{d}r|$; thus (1.5) integrates $\mu/(1-\mu)$ against $|\,\mathrm{d}r|/r$. For the charged system the exact expansion coefficient contains $\mu - Q^2/r^2$, which explains the signed form (1.2). The equivalence is proved below; it is not assumed as a cancellation-free interpretation of blueshift.

1.1.3 Christodoulou's censorship argument and its continuation hypothesis

All of Christodoulou's collapse, BV, and cosmic-censorship results discussed here concern the *real, massless, uncharged* Einstein–scalar-field system: $Q = q_0 = 0$ and ϕ is real. We separate his instability and global-structure arguments from the central continuation input on which *both* censorship conclusions depend. We then use Theorem 1.1 to supply that input in the AC class.

The characteristic datum on an outgoing cone $\mathcal{C}_0$ with regular vertex is

$$\alpha_0(r) = \partial_r(r\phi)|_{\mathcal{C}_0} = \frac{\alpha}{\lambda}\Big|_{\mathcal{C}_0}, \qquad \phi_0(r) = \frac{1}{r}\int_0^r \alpha_0(s)\,ds.$$

We use Christodoulou's space of AC data of finite total variation on $[0,\infty)$, with the regular-vertex constraints, and fix scalar translation by $\lim_{r\to\infty}\phi_0(r) = 0$. For BV data this is equivalent to $\lim_{r\to\infty}\alpha_0(r) = 0$. The topology is the one induced by

$$\|h\|_{\mathrm{BV}} = |h(0+)| + \mathrm{TV}_{[0,\infty)}\,h.$$

Here the value at the vertex is the right trace. This is the characteristic data framework in [10, pp. A32–A34], not a claim about genericity in a smooth-data topology. Completeness of $\mathcal{I}^+$ below has Christodoulou's affine-length meaning. A future-complete development is one that is timelike and null geodesically complete to the future.

Theorem 1.2 (Conditional consequence of Christodoulou's instability and global-structure results)**.** *In the preceding real uncharged AC characteristic framework, assume the following central continuation property at every candidate first singular endpoint, written as $(u,v) = (0,0)$:*

$$\int_{u_i}^{0}\frac{(-\nu)\mu}{(1-\mu)r}(u,0)\,du < \infty \quad\Longrightarrow\quad \textit{admissible central continuation in the development class.} \tag{1.6}$$

The past of the endpoint is regular and untrapped; continuation must agree with the original development by uniqueness.

Let $\mathcal{E}$ be the complement, in the preceding AC data class, of the data whose maximal future developments satisfy one of these two alternatives: the development is future complete; or $\mathcal{I}^+$ is complete and its singular future boundary consists of its initial central endpoint and a strictly spacelike $r = 0$ portion issuing from that endpoint, with no nontrivial central null boundary component.

For every $\alpha_0 \in \mathcal{E}$, there is a nonzero normalized AC function f, of finite total variation, such that every $\alpha_0 + tf$, $t \in \mathbb{R} \setminus \{0\}$, has an incomplete development satisfying the second alternative. The affine lines $\alpha_0 + \mathbb{R}f$ can be chosen pairwise disjoint for distinct exceptional data. Thus the exceptional set has codimension at least one in this perturbative sense, in the AC class with its induced BV topology.

Outside this same exceptional set, $\mathcal{I}^+$ is complete and the singular future boundary admits no C^2 regular future extension. For classical developments, the future-complete alternative and the curvature obstruction in the singular alternative give future C^2-inextendibility.

The attribution is to the implication obtained from [5, 6, 8], with the continuation hypothesis displayed; see Christodoulou's own AC formulation and proof outline in [10, pp. A33–A34], and [15, Theorems 1, 3 and Remark 1]. In [8, Lemma 2 and the following paragraph, pp. 190–191], finite terminal blueshift gives $\mu \to 0$ along the last cone. Continuation is then invoked to exclude this case at a singular endpoint, after which the instability argument uses divergent blueshift. Consequently, the global conclusions cannot be used here independently of (1.6). In particular, the exceptional set in [8, Section 4] already lies in the divergent-blueshift regime.

Corollary 1.3 (Uncharged AC weak and strong cosmic censorship from the extension theorem)**.** *Theorem 1.1, together with Christodoulou's instability, trapped-surface, and global-structure results, yields the following conclusions: For AC finite-total-variation initial data outside an exceptional set of codimension at least one (in the sense of Theorem 1.2), the following hold:*

1. Weak cosmic censorship: *future null infinity $\mathcal{I}^+$ is complete.*

2. Strong cosmic censorship in the C^2 future-boundary sense: *the development is future complete, or has the singular future boundary described in Theorem 1.2; this boundary admits no regular C^2 continuation. When the development's metric is C^2, this is future C^2-inextendibility as a Lorentzian manifold, without imposing spherical symmetry on a putative extension.*

The two conclusions use the same exceptional set. No terminal blueshift assumption is imposed on the data in this corollary.

Proof. We first check that our extension theorem applies in Christodoulou's characteristic framework. The local AC regularity, characteristic BV bounds, and regular-axis conditions on proper subsets are those supplied by the characteristic theory [6]. In the regular, untrapped past of a candidate first central endpoint, reparametrizing the null coordinates gives the triangular gauge and initial radial normalization of Theorem 1.1. These local properties also hold for the AC perturbations used in the instability argument.

If the terminal integral in (1.6) is finite, Theorem 1.1, applied with $Q = q_0 = 0$ and real ϕ, gives continuation through the endpoint in the local characteristic AC/BV class. Uniqueness on the overlap identifies this continuation with the original development. At a singular endpoint of the maximal development this contradicts maximality. Hence the terminal integral diverges at every remaining candidate endpoint. Its integrand is nonnegative, so this is the divergent-blueshift alternative required by the uncharged instability argument.

This verifies the continuation hypothesis of Theorem 1.2; it does not assume either censorship conclusion. Christodoulou's AC perturbation construction gives the disjoint exceptional lines. His trapped-surface and global-structure results then give the stated future-boundary alternatives and completeness of $\mathcal{I}^+$.

For the C^2 conclusion one uses curvature blow-up, not merely the spacelike character of the boundary. In the trapped region the invariant estimate

$$R_{abcd}R^{abcd} \geq \frac{4}{r^4}\left(\frac{2m}{r}\right)^2 \geq \frac{4}{r^4}$$

excludes a C^2 continuation through the spacelike $r = 0$ portion; see [19, Section 5.8, (69)]. At the initial central endpoint, the geometric extension argument in [19, Section 5.8.4, Lemma 1] gives a bound on this scalar invariant in a past diamond of any hypothetical C^2 extension, hence on a tail of the terminal cone. The first inequality then gives $\mu \leq Cr^2$ on that tail. After shrinking it so that $\mu \leq 1/2$,

$$\int_{\text{terminal tail}} \frac{\mu}{1-\mu}\frac{|\,\mathrm{d}r|}{r} \leq 2C\int_{\text{terminal tail}} r\,|\,\mathrm{d}r| < \infty.$$

The earlier compact cone segment contributes a finite integral by the proper-development hypotheses. The extension theorem therefore removes this central endpoint as well, contradicting its singularity.

The global boundary classification and the geometric reduction in [19, Section 5.8.4] identify the possible endpoints of a curve exiting an arbitrary C^2 extension; no symmetry of that extension is required. Compare also [28, Section 11, Lemmas 11.3–11.6]. In the future-complete alternative there is no finite future endpoint of a causal geodesic through which a C^2 extension could continue. □

For merely AC developments, the substantive SCC assertion is the geometric description of the future boundary and its obstruction to C^2 continuation; additional C^2 regularity of the original metric is not asserted, and initial roughness is not used as an inextendibility argument. The geometric extension argument just cited is used for this interpretation, not as an application of Luk–Oh's two-ended SCC theorem to collapse.

The BV two-parameter exceptional-set result of [8, Theorem 4.1] has its own continuation input in the BV framework. We do not claim to discharge it for arbitrary BV data: our corollary uses only the AC perturbation direction, as in [15, Remark 1]. Nor do we assert codimension two in AC, openness of the nonexceptional set, or genericity in a C^1 or smooth-data topology.

1.2 Previous work and the continuation argument

We now discuss the earlier work on which the proof builds, beginning with Christodoulou's characteristic and BV theories. We then explain how the extension criteria and instability results relate to the theorem proved here.

1.2.1 Characteristic evolution, mass concentration, and the BV theory

Characteristic evolution and generalized solutions. For the real, uncharged Einstein–scalar-field system, Christodoulou's characteristic formulation begins with data on an outgoing null cone with a regular vertex. His classical local existence and uniqueness theory and small-data dispersion [1] provide the basic setting in which a possible first central endpoint can be discussed. The generalized solutions of [2], and their structure and uniqueness in [3], concern a broader continuation of the evolution. Such generalized evolution should be distinguished from a regular extension through the missing center point: the latter must recover the axis conditions and identify the extended solution with the original development.

Mass concentration and singularity formation. The relation between mass concentration and singularity formation is made quantitative by Christodoulou's annular trapped-surface criterion [5]. His analysis of gravitational collapse [4] also relates positive final Bondi mass to black-hole formation and late-time geometry. The naked-singularity examples in [7] show that asymptotically flat characteristic data of finite regularity, even in spherical symmetry, do not by themselves rule out a singular center. A criterion excluding central concentration must therefore contain information beyond ordinary local existence.

The BV theory and the small-mass continuation criterion. The most direct analytic foundation here is Christodoulou's BV theory [6]. Its contributions relevant to this manuscript include control of the variation of the normalized characteristic scalar datum, the C^1 center continuation criterion involving small mass ratio throughout a past region [6, p. 1172, Theorem 5.1 and (5.1)], and global existence for sufficiently small BV data. The differentiated equation for α' and the interpolation between a radial derivative bound and the mass equation are also classical ingredients; see [6, p. 1173, (5.2), and p. 1175, (5.7)–(5.8)].

Characteristic variation and area estimates. The estimates needed to recover characteristic variation are related to the BV methods of Christodoulou [6] and the charged area estimates of An–Tan [20, Section 5]. Their common purpose is to control derivatives through integrated equations without requiring those derivatives to remain pointwise bounded. Section 2.6 explains how we use this idea and derives the charged identities. The characteristic variation estimates needed for local existence are proved in Section 3.3.

Local restart in double-null coordinates. Luk and Oh [12] provide a useful double-null implementation of the uncharged theory. Their Theorem 5.12 restates Christodoulou's small-BV theorem. The proof of their Theorem 5.14 normalizes the outgoing coordinate on a truncated cone, extends its datum, and uses finite propagation to apply the small-data result locally. This is a precise antecedent for the restart construction used here. Their Theorem 5.14 is itself a variation estimate in the locally BV-scattering setting, rather than a new general continuation criterion. For the charged problem, the corresponding local construction and uniqueness are supplied by Section 3; the uncharged theorem cannot simply be applied to a charged solution.

1.2.2 Charged evolution and extension away from the axis

For charged matter, Chae [18] established classical small-data global existence, decay, and completeness for the spherically symmetric Einstein–Maxwell–Higgs system by a contraction argument for a reduced evolution equation. This is an early small-data charged result, but its differentiability and decay assumptions should not be identified with small total variation of an AC characteristic datum. In particular, it does not provide the finite-cone theorem used in our restart.

Kommemi [19] gives the global boundary framework for spherically symmetric charged scalar field spacetimes and the reduced equations used here. His generalized extension principle, Theorem 1.8, is formulated for smooth initial data and applies to a characteristic diamond of finite quotient spacetime volume on which $0 < r_0 \leq r \leq R < \infty$. It can handle regions with trapped spheres and uses the null structure of the equations. The positive lower bound for r is essential: the theorem does not settle the limit $r \to 0$ at the center, which is the question addressed by Theorem 1.1.

The distinction also explains why the present local problem is not obtained by attaching a Maxwell equation to the real-scalar proof. Electric charge is sourced by the complex scalar current. Covariant transport carries a phase, and a constant translation of ϕ is no longer a symmetry when

$q_0 \neq 0$. A regular center requires $Q = 0$ there, whereas an inner boundary at positive radius can support an independent Coulomb charge.

1.2.3 Earlier central extension criteria

An and Tan [20] developed a charged gravitational-collapse analysis combining trapped-surface formation, scale-critical BV area estimates, instability arguments, and a global exceptional-set argument, following Christodoulou's programme in the uncharged case.

The charged uniform hypothesis is

$$\sup_{u_i \le u \le v < 0} \int_{u_i}^{u} \frac{(-\nu)(\mu - Q^2/r^2)}{(1-\mu)r}(s, v)\, \mathrm{d}s < \infty. \tag{1.7}$$

For the real uncharged system, this reduces to the same expression with $Q = 0$.

An, Chen, and Singh [21] proved a quantitative C^1 extension principle for the real uncharged scalar field under this uniform hypothesis and the additional restriction $\sup \mu < 3/8$.

An extension principle analogous to Theorem 1.1 was recently obtained by An–Chen–He [22] for both the real and charged systems, assuming C^1 initial data and the uniform blueshift bound (1.7), without a smallness assumption on μ. Already for C^1 data under the uniform-blueshift assumption, our proof relies on a key differential identity which, integrated along timelike curves, yields finite-cylinder dissipation and observability estimates; see Section 1.2.5. This differs from the successive limiting arguments of [22].

Theorem 1.1 allows AC normalized data and assumes a bound only on the terminal cone. In a charged instability argument, it excludes a nonextendible central endpoint with finite terminal blueshift. This continuation step does not replace An–Tan's charged trapping and perturbation arguments, whose hypotheses and data topology must also be matched. We refer to [22, Appendix A] for their treatment of the charged local theory. For completeness, we include in Section 3 a self-contained proof of the local AC theory in the precise form required for our restart argument, including the variation estimates and uniqueness.

A related mass-gap extension principle for the circularly symmetric Einstein–scalar-field system in $2+1$ dimensions is discussed in Section 1.3.2.

1.2.4 Instability: uniform versus terminal blueshift

It is important to distinguish the continuation input needed for the particular censorship argument in Section 1.1.3 from what suffices for a weaker formulation of weak cosmic censorship. For the real, uncharged Einstein–scalar-field system, Junbin Li's recent interior instability theorem [16, Theorem 1.3] allows one to prove WCC using only a continuation criterion under the uniform hypothesis

$$\sup_{u_i < u < v \le 0} \int_{u_i}^{u} \frac{(-\nu)\mu}{(1-\mu)r}(s, v)\, \mathrm{d}s < \infty. \tag{1.8}$$

This avoids the need for a continuation criterion under the weaker terminal hypothesis

$$\sup_{u_i < u < 0} \int_{u_i}^{u} \frac{(-\nu)\mu}{(1-\mu)r}(s, 0)\, \mathrm{d}s < \infty. \tag{1.9}$$

Moreover, for this purpose, a continuation theorem for C^1 data already suffices: approximation in the AC class, followed by instability and BV Cauchy stability, yields a Baire-category formulation of WCC. This argument does not require an AC terminal-cone continuation theorem, but it also does not establish SCC or recover Christodoulou's stronger description of the exceptional set.

Why uniform blueshift suffices for Li's argument. Consider first C^1 normalized characteristic data and a nonextendible first central endpoint, placed at $(0,0)$. Suppose that no trapped sphere has already formed. A continuation criterion under the uniform-blueshift assumption (1.8) implies, by contraposition, that there exists a sequence of preceding ingoing cones $v = v_n \uparrow 0$ such that

$$\int_{u_i}^{v_n} \frac{(-\nu)\mu}{(1-\mu)r}(u, v_n)\, du \longrightarrow +\infty.$$

Each of these cones extends from the initial outgoing cone to a regular point of the axis. This conclusion concerns a sequence of regular cones; it does not assert divergence of the integral on the terminal cone $v = 0$.

The relation to the incoming scalar flux can be seen directly from the uncharged equations. Since $\xi = r\partial_u\phi$,

$$\partial_u \log(1-\mu) = -\frac{(-\nu)\mu}{(1-\mu)r} + \frac{\xi^2}{(-\nu)r}.$$

Integrating to the regular axis, where $\mu \to 0$, gives

$$\int_{u_i}^{v_n} \frac{\xi^2}{(-\nu)r}(u, v_n)\, du = \int_{u_i}^{v_n} \frac{(-\nu)\mu}{(1-\mu)r}(u, v_n)\, du - \log\big(1 - \mu(u_i, v_n)\big).$$

The initial-cone term remains bounded near the terminal footpoint. Thus the preceding-cone blueshift divergence supplies precisely the divergent incoming flux used in [16, Theorem 1.3 and Remark 2.1]. Li's conclusion is that the development already contains a trapped sphere, or can be approximated by developments containing one, with perturbations that are AC and arbitrarily small in BV. A trapped sphere then implies completeness of future null infinity through the global results of [26]. The essential point is that Li's argument uses increasingly large blueshift on preceding regular cones, without requiring an infinite blueshift on the terminal cone itself.

Why C^1 continuation already gives a weaker WCC statement in the AC class. Here C^1 and AC refer to the normalized first-order characteristic datum α/λ, rather than merely to the scalar field ϕ. Fix the normalization at infinity and use the BV topology on this datum. For AC functions, convergence of the radial derivatives in L^1, together with the normalization, gives convergence in BV. Approximating these derivatives by smooth functions therefore gives the required density of C^1 data in the AC finite-variation class.

Let h denote an arbitrary normalized AC datum and choose C^1 data $h_k \to h$ in BV. If the development of h_k is future complete, retain h_k. If it already contains a trapped sphere, it has complete future null infinity and can again be retained. Otherwise, its first central singularity satisfies the preceding-cone blow-up condition by the C^1 uniform-blueshift continuation criterion. Li's theorem then supplies an AC perturbation $\widetilde{h}_k$ whose development contains a trapped sphere and such that

$$\|\widetilde{h}_k - h_k\|_{BV} < \frac{1}{k}.$$

Consequently,

$$\|\widetilde{h}_k - h\|_{BV} \le \|\widetilde{h}_k - h_k\|_{BV} + \|h_k - h\|_{BV} \longrightarrow 0.$$

This proves density of data whose developments are either future complete or contain a trapped sphere and have complete future null infinity. The continuation theorem has only been applied to the C^1 approximants; no continuation assertion for the original AC datum is needed.

Density alone is not a genericity statement. In the complete AC data space and BV Cauchy-stability framework used in [16, Section 1.4, Remark 1.7], the argument can be strengthened

as follows. Let $\mathcal{R}$ denote the data with future-complete developments and $\mathcal{G}$ the data whose developments contain a trapped sphere. The set $\mathcal{G}$ is open by Cauchy stability. Writing $\mathcal{R}_N$ for the open set of data whose central evolution remains regular beyond proper time N, the global existence framework gives

$$\mathcal{R} = \bigcap_{N=1}^{\infty} \mathcal{R}_N, \qquad \mathcal{R} \cup \mathcal{G} = \bigcap_{N=1}^{\infty} (\mathcal{R}_N \cup \mathcal{G}).$$

Since $\mathcal{R} \cup \mathcal{G}$ is dense, every open set $\mathcal{R}_N \cup \mathcal{G}$ is dense. Thus the data with complete future null infinity contain a dense G_δ set: WCC holds in this Baire-category sense. This formulation must be understood in the stated complete data space; additional asymptotic restrictions require the corresponding density and stability properties. It does not recover Christodoulou's disjoint affine perturbation lines through exceptional data or the resulting codimension statement.

Why this does not establish SCC. The distinction is geometric. A trapped sphere guarantees a black-hole region and, under the relevant global hypotheses, complete future null infinity. It does not determine the entire singular boundary inside the black hole. In particular, it does not exclude a locally naked central singularity with an outgoing Cauchy-horizon component $\mathcal{CH}_\Gamma$ contained inside that region. Such a component is compatible with completeness of $\mathcal{I}^+$, as is explicit in the global boundary classification of [19].

Li's perturbations alter the central evolution and may move its first singular endpoint. The conclusion that a trapped sphere forms does not imply that trapped spheres accumulate at the first central endpoint of the perturbed development, nor that $\mathcal{CH}_\Gamma$ is absent. Consequently, the set $\mathcal{G}$ used above is suitable for WCC but is not, by its definition, a set of developments known to satisfy SCC. An SCC argument must additionally exclude the remaining central components or establish inextendibility across them.

This explains the stronger geometric information sought in Christodoulou's exterior instability argument. The perturbation preserves the past of the relevant central endpoint, and trapping is arranged at scales approaching that endpoint; see [8, 15]. For the implementation discussed in Section 1.1.3, the terminal-cone extension criterion supplies the required alternative: a nonextendible central endpoint must have divergent terminal blueshift. Failure of a uniform bound on preceding cones does not, by itself, give that terminal-cone divergence. Li's theorem bridges this gap for trapped-sphere formation and WCC, but its stated conclusion does not provide the additional control of the central singular boundary needed for that SCC argument.

Liu–Li [15] provide important precedents through their double-null reformulation of exterior instability, their a priori estimates and concentration argument, and their AC perturbation direction [15, Remark 1]. The running-maximum scalar comparison used in the uniform-blueshift literature is also presented in Singh's dissertation [14, Chapter 7, Proposition 31], with attribution there to Liu–Li. The uniform-blueshift route to WCC described here concerns the uncharged system. A charged version requires the corresponding charged instability and solution-class arguments.

1.2.5 Boundary observation and related multiplier methods

The public note hosted by Cicortas [13] established an analogue of Theorem 1.1 for uncharged collapse, assuming C^1 normalized characteristic data and a uniform blueshift bound. Here we extend the argument to charged matter and AC data, and assume finite blueshift only on the last cone. As discussed above, the uniform-C^1 result already gives a weaker form of WCC; the AC terminal-cone theorem allows us to apply Christodoulou's argument as in Corollary 1.3.

A key ingredient in the proof of [13] is a new boundary balance identity along a timelike curve $\{r = b\}$. This identity persists in the charged case. With T denoting differentiation along $r = b$ using the original parameter u, we have

$$Q_b = \frac{b}{2}\left\{\left|\frac{\theta}{\lambda}\right|_b^2 - \log(1-\mu_b)\right\} \geq 0, \tag{1.10}$$

$$\frac{\mathrm{d}Q_b}{\mathrm{d}u} + \frac{b^2}{2(-\nu_b)}|D_T\phi|_b^2 = b\ \mathrm{Re}\left(\frac{\overline{\theta}}{\lambda}\, D_T(\alpha/\lambda)\right)_b. \tag{1.11}$$

Section 2.4 derives this identity, including the cancellation of the Maxwell contribution to the tangential mass flux. The identity alone is insufficient: the energy could remain positive while its dissipation were small. The other key ingredient is an observation estimate for the wave equation, which relates the energy to the dissipation on $r = b$ and the variation of α/λ along that curve. This explains why the differentiated scalar transport equation is already needed in the argument showing that μ tends to zero.

This step is crucial, but the precise estimate matters. Under the geometric and smallness assumptions of Proposition 2.10, we prove

$$b\left\{\left|\frac{\theta}{\lambda}\right|_b^2(\tau_0) + \mu_b(\tau_0)^2\right\} \leq C_*\left\{\int_I \left|\frac{\theta}{\lambda} - \frac{\xi}{\nu}\right|_b^2 \mathrm{d}\tau + b\left[\mathrm{TV}_I^D((\alpha/\lambda)_b)\right]^2\right\},$$

where $\mathrm{d}\tau = (-\nu_b)\,\mathrm{d}u$ and I is an observation interval of length comparable to b centered at τ_0. Notice that the mass term on the left is μ_b^2, whereas Q_b/b contains $-\log(1-\mu_b)$, which is comparable to μ_b when μ_b is small. We therefore need a longer interval to obtain a definite decrease of Q_b. More precisely, Proposition 2.11 shows that, if $Q_b(0)/b \leq \varepsilon$, then

$$\frac{Q_b(H_d b/\varepsilon)}{b} \leq d\varepsilon + C_d\left[\mathrm{TV}^D_{[0,H_d b/\varepsilon]}((\alpha/\lambda)_b)\right]^2, \qquad 0 < d < 1,$$

provided the hypotheses on the cylinder hold throughout that interval. Thus we decrease a prescribed upper bound by a fixed fraction, up to the trace-variation error, on an interval of length $H_d b/\varepsilon$. This is not a contraction of every actual initial energy on an interval of length Cb independent of its size. In the argument below, we must both find a cylinder that lasts long enough and control the variation of α/λ on it.

The argument is related in structure to Morawetz and r^p estimates. An energy balance such as (1.11) alone does not imply decay: one also needs an estimate ensuring that persistent energy produces a definite amount of flux or dissipation. Dafermos–Rodnianski's physical-space method [23] implements this principle by combining weighted energy identities, integrated local energy decay, and iteration in time. Here the timelike-curve identity gives the balance law, and the observation estimate relates central concentration to boundary dissipation and the variation of the boundary trace. Integrating over overlapping time intervals then gives the reduction above. We study concentration at small radius, with an energy containing the coupled term $-\log(1-\mu)$; the balance identity is not a particular member of the r^p hierarchy.

There is a similar connection with the nonconcentration argument of Christodoulou–Tahvildar-Zadeh for spherical wave maps [17]. Their argument combines estimates controlling kinetic energy with a separate use of the target geometry to control spatial energy. This strategy motivated the search for a nonconcentration estimate in [13, Section 1.4]. In the present argument, the positive term in the timelike-curve identity controls tangential scalar variation, but does not by itself control all the energy represented by Q_b. The observation estimate gives the additional information

needed to exclude persistent concentration with insufficient dissipation. The exact Einstein–scalar cancellation in (1.11) is specific to the coupled equations, but this use of partial energy control together with a geometric estimate has a clear precedent in the wave-map work.

1.2.6 The three stages of the proof

We develop the argument in three stages:

$$\begin{aligned} &C^1 \text{ data, uniform blueshift} \\ &\longrightarrow \text{AC data, uniform blueshift} \\ &\longrightarrow \text{AC data, terminal-cone blueshift.} \end{aligned}$$

The passage from C^1 to AC data requires us to estimate α' without a pointwise bound. The last step requires us to control the interior using information on the terminal cone alone. We give one detailed charged proof below, for AC data with finite terminal blueshift; it also covers the case of a uniform blueshift bound.

First stage: C^1 data and a uniform blueshift bound. The uniform hypothesis (1.7) gives a lower bound for λ through its transport equation. The upper bound follows separately from Raychaudhuri:

$$\partial_u \log \lambda = -\frac{(-\nu)(\mu - Q^2/r^2)}{(1-\mu)r}, \qquad \partial_u \log \frac{\lambda}{1-\mu} = -\frac{|\xi|^2}{(-\nu)r} \le 0.$$

Together with the positive initial gap, these identities give $0 < c \le \lambda \le C$; the second calculation is also used in Proposition 2.3. The new boundary balance identity, observation estimates, and transport of the boundary trace then exclude central concentration, once we have found a small cylinder that lasts long enough. This proves that μ tends to zero uniformly throughout the past triangle:

$$\lim_{a \uparrow 0} \sup_{a \le u < v \le 0} \mu(u,v) = 0. \tag{1.12}$$

To conclude in the uncharged case, we apply Christodoulou's C^1 extension criterion [6]: sufficiently small μ throughout a past neighborhood of the central endpoint ensures continuation through that endpoint. The uniform decay in (1.12) gives precisely this smallness after restricting to a sufficiently small neighborhood. In the charged case, we obtain small characteristic variation and apply the local theory of Section 3. To preserve C^1 regularity, we prolong the selected datum in C^1 with small added variation, as described in Section 3.6.

Remark 1.4 (Vanishing mass ratio is insufficient for AC data: the uncharged case). Already for the real uncharged Einstein–scalar-field system, $\mu(u,0) \to 0$ along the terminal ingoing cone does not suffice for regular-center BV continuation from AC characteristic data. The counterexample in [9, Theorem 1.1] has an AC normalized first-order datum α/λ of finite total variation and a regular center on every proper past triangle, but no regular-center BV extension through the terminal endpoint. In fact, after translating that endpoint to $(0,0)$, it satisfies the stronger decay condition

$$\lim_{a \uparrow 0} \sup_{a \le u < v \le 0} \mu(u,v) = 0, \qquad \int_{u_i}^{0} \frac{(-\nu)\mu}{(1-\mu)r}(u,0)\, du = +\infty.$$

Thus even uniform vanishing of the mass ratio throughout the shrinking causal past cannot replace finite terminal blueshift in (1.6) at AC regularity. The example violates the finite-blueshift hypothesis of our theorem. The regularity distinction concerns the normalized first-order datum: the initial scalar field in the construction is C^1, whereas α/λ has no positive Hölder exponent on any neighborhood of the distinguished initial sphere.

Second stage: AC data with the same uniform hypothesis. The outgoing radial derivative α' of α/λ is now initially only in L^1. The heart of the argument is to separate the part coming from the initial data from the part generated by the transport equation. On a proper positive-radius rectangle with initial cone $u = u_0$, Lemma 2.1 gives

$$\alpha'(u,v) = \frac{\lambda(u_0,v)^2}{\lambda(u,v)^2} \exp\left(-iq_0 \int_{u_0}^{u} A_u(s,v)\,\mathrm{d}s\right) \alpha'(u_0,v) + \alpha'_{\mathrm{rem}}(u,v).$$

Here α'_{rem} is the source integral with its integrating factor in (2.19), divided by $\lambda(u,v)$ as in (2.4). The factor involving λ is bounded under the uniform hypothesis, and the phase has modulus one. Thus we can estimate the part coming from the initial data by its L^1 integral on the relevant interval of v-labels. The remainder α'_{rem} is estimated using the geometric and scalar bounds, as in the C^1 argument. This decomposition allows us both to prove (1.12) and to control the variation of α/λ on the timelike curves.

Unlike in the C^1 case, smallness of μ does not yet give continuation. We still have to prove that the variation of α/λ is small on a suitable outgoing cone. For this, we introduce the renormalized quantities in Section 2.6, which keep the cubic scalar and complex curvature terms together. In particular, the definition of j in (2.95) gives the exact identity

$$\partial_v(\alpha/\lambda) = \frac{j}{1-\mu} + \frac{\lambda}{r}\left|\frac{\theta}{\lambda}\right|^2 \frac{\theta}{\lambda}.$$

The cubic term is controlled through the radial identity

$$r\frac{\theta}{\lambda}(u,r) = b\frac{\theta}{\lambda}(u,b) + \int_b^r x\partial_x(\alpha/\lambda)(u,x)\,\mathrm{d}x.$$

In particular, if $|\theta/\lambda| \le \delta < 1$ on the row, then

$$(1-\delta^2)\,\mathrm{TV}_{[b,R]}(\alpha/\lambda) \le \int_{v_b(u)}^{0} \frac{|j|}{1-\mu}\,\mathrm{d}v + \delta^2\left|\frac{\theta}{\lambda}\right|_b,$$

where $r(u, v_b(u)) = b$ and $R = r(u,0)$. The smallness of θ/λ therefore allows us to absorb the cubic term. It remains to estimate the integral of $|j|/(1-\mu)$.

We do this using the coupled equations for j and the full complex curvature quantity $\mathcal{C}$. We first express $r\mathcal{C}$ as a characteristic integral, substitute this expression into the equation for j, and integrate in the appropriate order. This gives estimates for the derivative source and the boundary variation. Here we use the characteristic variation and area-estimate methods discussed above. The two estimates are proved together in (2.134). Passing to the limit of the shrinking cylinders then gives (2.136), which controls the variation on a complete cone. By taking the smallness parameters to zero and choosing the initial row sufficiently late, we obtain a sequence of complete cones $u = u_n$, with $u_n \uparrow 0$, such that

$$\mathrm{TV}_{[u_n,0]}\left(\frac{\alpha}{\lambda}(u_n,\cdot)\right) \longrightarrow 0.$$

The charged length scale is also small on these cones. We conclude by applying the local theory for AC data of small total variation in Section 3. Uniqueness on the overlap identifies the solution constructed there with the original solution, so this gives the required continuation.

Third stage: AC data and only terminal-cone blueshift. The new difficulty is to control the interior of the past triangle using information on the terminal cone alone. A direct comparison with that cone is effective where the area radii are comparable, but loses control near the axis. Even in the uncharged case, mass monotonicity gives only

$$\mu(u,v) \le \mu(u,0)\frac{r(u,0)}{r(u,v)},$$

whose last factor is unbounded as the axis is approached. Neither this inequality nor the terminal mass-ratio estimate in [8, Lemma 2] controls the remaining central region.

We first express the terminal hypothesis as finiteness of a positive integral and deduce $\mu(u,0) \to 0$ (Proposition 2.3). For each fixed $0 < \epsilon < 1$, define

$$\mathcal{D}_\epsilon = \{(u,v) \in \mathcal{Q} : u < v \le 0, \quad r(u,v) \ge \epsilon r(u,0)\}.$$

Mass monotonicity then gives

$$\sup_{\{v:(u,v)\in\mathcal{D}_\epsilon\}} \mu(u,v) \le \frac{\mu(u,0)}{\epsilon} \longrightarrow 0 \qquad \text{as } u \uparrow 0.$$

This comparison also holds in the charged case, since m is the ordinary Hawking mass and $\partial_v m \ge 0$. However, the estimate is not uniform as $\epsilon \downarrow 0$, so it leaves open the possibility of concentration at radii much smaller than $r(u,0)$.

To go further, we choose an exterior region whose inner radius is a progressively smaller fraction of $r(u,0)$. Proposition 2.5 gives small mass ratio, small $|\theta/\lambda|$, and small characteristic variation in this region, together with bounds on the null expansions.

The uniform-AC case follows as a special case. At each fixed $u < 0$, continuity allows us to pass to the limit $v \to 0$ in the truncated integrals of (1.7), giving (1.2). This passage takes place away from the central endpoint and remains valid for the signed charged integrand.

1.3 Related problems and the scope of the result

1.3.1 Regularity and the topology of perturbations

Throughout this paper, AC refers to the normalized outgoing datum α/λ, rather than merely to ϕ. In the initial radial and electromagnetic gauge,

$$\mathrm{TV}_{[0,R]}(\alpha/\lambda) = \int_0^R |\alpha'(r)| \, \mathrm{d}r.$$

This distinction matters when comparing continuation, instability, and stability results. All the naked-singularity results discussed below concern the real, uncharged Einstein–scalar-field system.

Christodoulou's continuously self-similar examples [7] have scalar regularity C^{1,σ_k} at the initial sphere $r = r_*$ where the singularity's past light cone meets the data, with

$$\sigma_k = \frac{k^2}{1-k^2}, \qquad 0 < k^2 < \frac{1}{3}.$$

Their profile estimates imply

$$\partial_r(\alpha_k/\lambda_k) = 2\partial_r\phi_k + r\partial_r^2\phi_k = O\Big(1 + |r - r_*|^{\sigma_k - 1}\Big);$$

see [35, Proposition 2.1]. Thus the normalized datum is AC and C^{0,σ_k}, but generally not C^1. Its absolute continuity follows from this integrable derivative bound, not from Hölder regularity alone.

Christodoulou's instability construction [8, Section 4] uses two perturbation directions, one with a jump and one absolutely continuous. The latter gives perturbations within AC that are arbitrarily small in the BV topology. By contrast, Zheng's nonlinear stability theorem [34], building on Singh–Zheng's linearized analysis [35], proves persistence of naked singularities for sufficiently small nonzero k under small *scalar* perturbations in the localized spaces $\overline{\mathcal{C}}_N^{1/(1-k^2),\delta}$ specified there. These norms control weighted higher derivatives near r_* as well as exterior behavior. Smallness in BV does not imply smallness in these norms, so the stability and instability conclusions are compatible. Singh's earlier linear and nonlinear work [25, 24] also illustrates this dependence on regularity and on the permitted perturbations.

These examples do not contradict our extension criterion: on their terminal cone,

$$\mu = \frac{k^2}{1+k^2}, \qquad \int_\rho^R \frac{\mu}{1-\mu}\frac{\mathrm{d}r}{r} = k^2 \log\frac{R}{\rho} \longrightarrow +\infty \quad \text{as } \rho \downarrow 0.$$

Their blue-shift factor is therefore unbounded.

1.3.2 Weak Cosmic Censorship and the mass gap in $2+1$ dimensions

For the circularly symmetric Einstein–scalar-field system with $\Lambda < 0$, Cicortas [11, Theorem 1.1] proved the following version of the weak cosmic censorship conjecture: for every integer $k \geq 2$, there is an open and dense subset of the C^k asymptotically AdS characteristic data space whose maximal developments contain no globally naked singularities. Reflecting boundary conditions are imposed at null infinity, and genericity is understood in the weighted Sobolev topology specified there. The proof combines a mass-gap extension principle, trapped-surface formation, and a blueshift instability implemented by perturbations within the C^k class.

The extension principle at the center follows from the mass-gap theorem [11, Theorem 1.2]. For C^k characteristic data with $k \geq 2$, the theorem rules out singularity formation in the domain of dependence of a finite outgoing cone issuing from a regular center, provided its outermost circle has Hawking mass strictly less than 1. The Hawking mass in the setting of [11] has the normalization

$$m_{2+1} = 1 - g(\nabla r, \nabla r) - \Lambda r^2.$$

At a first central singularity, $m_{2+1} \to 1$ along the terminal cone, and the terminal blueshift diverges [11, Corollary 1.1 and Proposition 5.1]. This explains the role of the extension principle in the censorship argument: it places every remaining central singularity in the divergent-blueshift regime required for instability. Theorem 1.1 supplies the corresponding continuation input in the present setting, under the terminal-cone hypothesis and at AC regularity. We also note that there is no analogous gap for the dimensionless mass ratio $\mu = 2m/r$ in the AC class in $3+1$ dimensions: Christodoulou's naked-singularity examples [7] have arbitrarily small positive limiting values of μ along the terminal cone.

1.3.3 Black-hole interiors and charged gravitational collapse

Uncharged gravitational collapse. In Christodoulou's *uncharged gravitational-collapse* model, the scalar field is real and massless, the Maxwell field vanishes, and the initial data have a regular center and one asymptotically flat end [5, 8, 10]. This is the setting of Corollary 1.3. Other results relevant to this regular-center setting include Dafermos's trapped-surface theorem [26], which relates trapping to completeness of future null infinity under its extension and geometric hypotheses, and the dispersive real-scalar-field analysis of Luk–Oh [12].

Two-ended charged black holes. A different problem concerns Einstein–Maxwell coupled to a *real, electrically neutral scalar field*, with nonzero source-free electromagnetic charge. Dafermos's work [27] supplies characteristic black-hole-interior results in this model, while the strong cosmic censorship theorems of Luk–Oh [28, 29] concern two-ended asymptotically flat Cauchy data. These results address Cauchy horizons and their singular behavior in a setting distinct from Christodoulou's regular-center collapse problem. Indeed, spherical source-free Maxwell charge is constant, whereas regularity at a center forces it to vanish. Consequently, the nonzero-charge model cannot describe collapse from a regular center without charged matter. The central continuation criterion proved here is therefore not an input to these two-ended results, whose developments have no regular center.

Charged gravitational collapse. For charged gravitational collapse, the scalar current allows the enclosed charge Q to evolve while $Q = 0$ at the regular center, permitting one-ended collapse with a charged black hole. The global geometric framework for this problem was established by Kommemi [19]. For smooth, one-ended asymptotically flat initial data without anti-trapped spheres, his Theorem 1.1 classifies the possible future boundary components, distinguishing in particular the central Cauchy horizon $\mathcal{CH}_\Gamma$ from the Cauchy horizon $\mathcal{CH}_{i^+}$ emanating from timelike infinity.

Moreover, [19, Theorem 1.1, VII.4] shows that any future C^2 extension entails an exiting timelike curve whose projected closure meets $\mathcal{CH}_\Gamma \cup \mathcal{CH}_{i^+}$. This identifies the central obstruction and the black-hole Cauchy horizon as the two issues to address for strong cosmic censorship; the classification itself does not establish their generic exclusion or singularity. The subsequent work of Van de Moortel [30, 31] analyzes $\mathcal{CH}_{i^+}$ under quantitative event-horizon hypotheses. In particular, [31, Theorem C] gives future C^2-inextendibility for the smooth one-ended developments considered there, under its exterior and subextremality assumptions and the exclusion of $\mathcal{CH}_\Gamma$. Our Theorem 1.1 concerns the complementary central problem: within its solution class and geometric hypotheses, a central endpoint with bounded terminal blue-shift factor admits continuation in the local characteristic AC/BV class. Consequently, an endpoint nonextendible in that class and satisfying the remaining hypotheses must obey

$$\sup_{u_i<u<0} \int_{u_i}^{u} \frac{(-\nu)(\mu - Q^2/r^2)}{(1-\mu)r}(s,0)\, ds = +\infty. \tag{1.13}$$

Generically excluding the remaining central endpoints requires a charged instability and trapping argument in a compatible data topology. Completeness of future null infinity alone does not exclude locally naked singularities inside a black hole. Our theorem gives the continuation result needed to address these central endpoints in Kommemi's classification; it does not by itself establish generic charged SCC. To combine it with the smooth global-structure and interior theorems in an AC/BV genericity argument, one must also justify applying those theorems at the regularity under consideration.

Organization of the proof. We prove the terminal-cone extension theorem in Section 2. After deriving the geometric bounds from the terminal-cone assumption, we prove the boundary observation estimate in Section 2.4 and the variation estimates in Section 2.6. We then use these estimates to continue the solution in Section 2.7. The local existence, stability, and uniqueness results for charged AC data used in this last step are proved in Section 3.

2 The charged terminal-cone proof

We use the fields, gauge, and solution class fixed in Theorem 1.1.

2.1 Reduced equations and local AC calculus

Besides the mass and Maxwell equations written in the next section, we use

$$D_u\theta = -\frac{\lambda}{r}\xi - \frac{iq_0\nu\lambda Q}{(1-\mu)r}\phi, \qquad D_v\xi = -\frac{\nu}{r}\theta + \frac{iq_0\nu\lambda Q}{(1-\mu)r}\phi, \tag{2.1}$$

$$\partial_v A_u = \frac{2\nu\lambda Q}{(1-\mu)r^2}, \qquad [D_u, D_v] = \frac{2iq_0(-\nu)\lambda Q}{(1-\mu)r^2}, \tag{2.2}$$

$$\partial_u \log\frac{\lambda}{1-\mu} = \frac{|\xi|^2}{\nu r}, \qquad \partial_v \log\frac{-\nu}{1-\mu} = \frac{|\theta|^2}{\lambda r}. \tag{2.3}$$

The commutator acts by multiplication on fields with the same charge as ϕ. A covariant derivative of a gauge-invariant scalar is its ordinary derivative.

Lemma 2.1 (An outgoing density with a common representative)**.** *Let $[a,b]\times[c,v_+]$ be a proper characteristic rectangle with positive lower bounds for r, λ, $-\nu$, and $1-\mu$. There is a measurable representative of $\alpha' = \lambda^{-1}\partial_v(\alpha/\lambda)$ with the following properties. For every v, the function $\alpha'(\cdot,v)$ is AC, and $\lambda\alpha'$ solves (2.19). There is one null set of labels, independent of u, outside which $\lambda(u,v)\alpha'(u,v) = \partial_v(\alpha/\lambda)(u,v)$ on every row. Moreover, on the rectangle,*

$$\lambda(u,v)\alpha'(u,v) = \frac{\lambda(a,v)}{\lambda(u,v)}\exp\left(-iq_0\int_a^u A_u(s,v)\,\mathrm{d}s\right)\partial_v(\alpha/\lambda)(a,v) + R(u,v), \tag{2.4}$$

where R is continuous and $R(a,v)=0$.

Proof. The undifferentiated incoming equation, already an integral equation in the solution class, is

$$\partial_u(\alpha/\lambda) = c\,(\alpha/\lambda) + f, \qquad c = \frac{-\nu}{r}\frac{\mu - Q^2/r^2}{1-\mu} - iq_0A_u, \qquad f = \frac{-\nu}{(1-\mu)r}\left[-\mu + \frac{Q^2}{r^2} + iq_0Q\right]\phi.$$

Both coefficients are continuous. More is available in the outgoing direction: $r_v = \lambda$, $\phi_v = \theta/r$, the mass and Maxwell equations, the equation for ν_v, and the constraint for $(A_u)_v$ have continuous right-hand sides on this rectangle. Hence c_v and f_v exist and are continuous. In particular this statement uses no outgoing derivative of λ and no derivative of α/λ.

Define the integrating factor

$$P(u,s;v) = \exp\left(\int_s^u c(t,v)\,\mathrm{d}t\right).$$

It and its v-derivative are continuous, and the incoming integral equation gives, for every (u,v),

$$\frac{\alpha}{\lambda}(u,v) = P(u,a;v)\frac{\alpha}{\lambda}(a,v) + \int_a^u P(u,s;v)f(s,v)\,\mathrm{d}s.$$

Differentiation of this formula in v is justified by the ordinary one-dimensional AC product rule and differentiation under an integral of continuous functions. Thus every row is AC, and its derivative satisfies

$$\partial_u(\lambda\alpha') = c\,\lambda\alpha' + c_v\,\frac{\alpha}{\lambda} + f_v$$

for almost every initial label. Equivalently,

$$\lambda(u,v)\alpha'(u,v) = P(u,a;v)g(v) + \int_a^u P(u,s;v)\left(c_v\frac{\alpha}{\lambda} + f_v\right)(s,v)\,\mathrm{d}s,$$

where g is a measurable representative of the initial L^1 derivative, assigned the value zero on its exceptional set. Division by the positive incoming-C^1 function λ defines α' for all labels and shows that each incoming section is AC. The integral term is continuous.

To obtain one exceptional set for all rows, take the non-Lebesgue points of g, together with its null set of nonfinite values. At every other v, the derivative of the AC initial datum exists and equals $g(v)$. The preceding formula for α/λ, whose other factors are C^1 in v, is consequently differentiable at that same v for every u. This proves the asserted simultaneous identification.

Finally,

$$\partial_u \log \lambda = -\frac{-\nu}{r}\frac{\mu - Q^2/r^2}{1-\mu},$$

so $P(u,a;v)$ has exactly the form in (2.4). Expanding $c_v(\alpha/\lambda) + f_v$ gives the continuous source in (2.19). In this expansion the term $-iq_0(A_u)_v(\alpha/\lambda)$ is retained; it is the electromagnetic commutator term. For example, with the ratios left explicit, the coefficient calculations use

$$\partial_v \log \frac{-\nu}{(1-\mu)r} = \frac{\lambda}{r}\left(\left|\frac{\theta}{\lambda}\right|^2 - 1\right), \qquad Q_v = -q_0 r \lambda \operatorname{Im}(\bar{\phi}\,\theta/\lambda).$$

These yield the $3Q\,\theta/\lambda$ and $Q\phi(1+|\theta/\lambda|^2)$ terms in that source. □

Lemma 2.2 (Timelike chain rule and exit identity)**.** *On a proper positive-radius region, α/λ has an AC restriction to every graph $u \mapsto (u, v(u))$ for which v is monotone and AC. Along such a graph,*

$$D_{\partial_u + v'\partial_v}(\alpha/\lambda) = D_u(\alpha/\lambda) + v'\lambda(u,v(u))\alpha'(u,v(u)) \tag{2.5}$$

almost everywhere. In particular this holds on the adaptive frontier and on constant-radius cylinders.

Proof. It suffices first to work in the rectangle of Lemma 2.1. Its representation and the bounded continuous incoming derivative imply

$$\left|\frac{\alpha}{\lambda}(u,v) - \frac{\alpha}{\lambda}(\widetilde{u},\widetilde{v})\right| \le C|u-\widetilde{u}| + C\left(|v-\widetilde{v}| + \int_{\min(v,\widetilde{v})}^{\max(v,\widetilde{v})} |g(s)|\,\mathrm{d}s\right).$$

Consequently the field is jointly Lipschitz after the strictly increasing AC change of label

$$z(v) = v + \int_c^v |g(s)|\,\mathrm{d}s.$$

Its inverse is Lipschitz. If $v(u)$ is monotone and AC, then $z(v(u))$ is AC: it is monotone, continuous, and maps null sets to null sets, because both constituent maps are AC. The displayed bound proves AC of the restriction.

For completeness the derivative can be identified without assuming a trace theorem for a two-dimensional weak derivative. Split a difference of the restricted field into a change in u at fixed label and a change in v on a fixed row. The first difference quotient converges to the continuous incoming derivative. At labels outside the common exceptional set, the second converges to $v'\lambda\alpha'$. The one-dimensional change-of-variables formula shows that $v' = 0$ almost everywhere on the preimage of that exceptional set. At these remaining points the second quotient vanishes almost everywhere, by the bound in terms of $z(v(u))$ and $(z\circ v)' = (1+|g(v)|)v'$ almost everywhere. This proves the ordinary chain rule, and addition of $iq_0A_u(\alpha/\lambda)$ gives (2.5). Finitely many rectangle charts cover a proper curve segment; their representatives agree off a common null set of labels and give the same restricted field.

Here is the exit calculation used subsequently. Suppose $v_-(u)$ is increasing and locally bi-Lipschitz on $[a,b]$, and write the density equation as $D_u(\lambda\alpha') = h\lambda\alpha' + s$, where $h = (-\nu)(\mu - Q^2/r^2)/((1-\mu)r)$ is real and s is its continuous source. For almost every label,

$$\partial_u(\lambda|\alpha'|) \le h\lambda|\alpha'| + |s|.$$

Integrate this inequality until the label exits through $v = v_-(u)$, or until the final row if it does not exit. Tonelli and the change of variables $v = v_-(u)$ give

$$\begin{aligned}\int_{v_-(b)}^{v_+} \lambda(b,v)|\alpha'(b,v)|\,\mathrm{d}v + \int_a^b v_-'(u)\lambda(u,v_-(u))|\alpha'(u,v_-(u))|\,\mathrm{d}u \\ \le \int_{v_-(a)}^{v_+} \lambda(a,v)|\alpha'(a,v)|\,\mathrm{d}v + \int_a^b \int_{v_-(u)}^{v_+} (h\lambda|\alpha'| + |s|)\,\mathrm{d}v\,\mathrm{d}u. \end{aligned} \tag{2.6}$$

All integrals are finite on a proper truncation, because $|\alpha'| \le C(1+|g(v)|)$. The left-hand boundary term is therefore an actual integrable trace, not an unspecified Sobolev representative. Nonnegative majorants allow exhaustion toward a terminal endpoint after the uniform estimates have been proved. □

Renormalized equations in the AC class. The preceding lemmas also justify Lemma 2.13 at the stated regularity. Indeed

$$j = (1-\mu)\lambda\left(\alpha' - \frac{1}{r}\left|\frac{\theta}{\lambda}\right|^2\frac{\theta}{\lambda}\right).$$

For every label outside the common null set, all factors in this expression are AC in u, and their incoming derivatives are the continuous reduced equations and the density transport. The ordinary product rule therefore gives (2.97) and (2.99) on each such incoming section. On every outgoing row, $\theta/\lambda = \alpha/\lambda - \phi$ is AC, while the outgoing equations for ξ/ν, μ, and r have continuous right-hand sides. Their AC product rule gives (2.98). The sources are bounded on the rectangle by $C(1+|g(v)|)$, so all these equations hold as characteristic integral identities and as distributional identities. The weighted modulus inequalities and their exit versions follow exactly as in (2.6). This argument requires neither an L^2 norm of g nor an incoming derivative of g.

For a curve with tangent T, covariant variation of a charged field is the ordinary variation after unitary parallel transport, equivalently $\int |D_T f|$ when the restriction is AC. In $A_v = 0$, outgoing variation is ordinary variation. A residual gauge depending only on u multiplies each outgoing row by one constant unit phase; it preserves moduli and outgoing variation, but preserves its complex datum only up to that phase. It can arrange $A(T) = 0$ on a selected cylinder or frontier. This gauge operation and the chain rule above do not change any of the stated variation estimates.

2.2 Charged terminal estimates and an adaptive exterior

Here $\mu = 2m/r$ uses the ordinary Hawking mass, Q is electric charge, and the scalar is complex and massless. We use $A_v = 0$, $D_u = \partial_u + iq_0A_u$, $\theta = r\partial_v\phi$, $\xi = rD_u\phi$, and $\alpha = \lambda\phi + \theta$. The initial incoming label is normalized by $\lambda(u_i, v) = 1/2$; the regular axis is $u = v$. The domain is $u_i \le u \le v \le 0$, with the endpoint $(0,0)$ removed. We assume $\lambda > 0$, $\nu < 0$, $0 \le \mu < 1$, the regular-axis conditions $r = m = Q = 0$, and $(\alpha/\lambda)(u_i, \cdot) \in \mathrm{AC}[u_i, 0]$. All statements below are a priori statements for the original local AC development: on every proper triangle the normalized scalar datum has finite outgoing variation, the undifferentiated fields are continuous, and the reduced equations hold in their integral form. These are the local solution properties, not additional uniform estimates at the missing endpoint. No interior gap or lower expansion bound is assumed.

2.2.1 The signed and positive terminal conditions are equivalent

Proposition 2.3 (Terminal hypotheses)**.** *Suppose the terminal blue-shift factor is bounded, equivalently that its logarithm satisfies:*

$$\sup_{u_i<u<0}\int_{u_i}^{u}\frac{-\nu}{r}\frac{\mu-Q^2/r^2}{1-\mu}(s,0)\,\mathrm{d}s<\infty. \tag{2.7}$$

Then the positive terminal integral is finite:

$$\int_{u_i}^{0}\frac{-\nu}{r}\frac{\mu}{1-\mu}(u,0)\,\mathrm{d}u<\infty. \tag{2.8}$$

Conversely, (2.8) implies (2.7). In particular the natural signed condition is sufficient for all the conclusions of this section.

Writing $R=r(u,0)$, $M(R)=m(u,0)$, and $R_i=r(u_i,0)=-u_i/2$, one has

$$R\longrightarrow 0,\qquad \frac{M(R)}{R}\longrightarrow 0,\qquad \int_0^{R_i}\frac{M(R)}{R^2}\,\mathrm{d}R<\infty,\qquad \lambda(u,0)\longrightarrow\lambda_0>0.$$

The terminal scalar flux is finite, and

$$|\phi(u,0)|\le C\left(1+\sqrt{\log(R_i/R)}\right). \tag{2.9}$$

More precisely, if ξ/ν is regarded as a function of R on the terminal cone, then

$$|Q(R)|^2\le Cq_0^2R^4\left(1+\log(R_i/R)\right)\int_0^R\left|\frac{\xi}{\nu}\right|^2\frac{\mathrm{d}s}{s}. \tag{2.10}$$

The constant can depend on the initial data and on the bound in (2.7).

Proof. The null mass equations are

$$m_v=\frac{(1-\mu)|\theta|^2}{2\lambda}+\frac{Q^2\lambda}{2r^2},\qquad m_u=\frac{(1-\mu)|\xi|^2}{2\nu}+\frac{Q^2\nu}{2r^2}.$$

Hence $m\ge 0$, $m_v\ge 0$, and $m_u\le 0$. Raychaudhuri gives

$$\partial_u\log\frac{\lambda}{1-\mu}=-\frac{|\xi|^2}{(-\nu)r}.$$

The positive initial gap therefore yields a global upper bound $\lambda/(1-\mu)\le C$, and in particular $\lambda\le C$. Integration from the existing axis proves $r(u,v)\le C(v-u)$, so $R\to 0$.

Let $\gamma(u)$ denote the signed integral in (2.7), and let $\delta_i=(1-\mu)(u_i,0)>0$. The equation for λ and Raychaudhuri give the exact identity

$$\gamma(u)=\int_{u_i}^{u}\frac{|\xi|^2}{(-\nu)r}(s,0)\,\mathrm{d}s-\log(1-\mu(u,0))+\log\delta_i. \tag{2.11}$$

Both terms preceding $\log\delta_i$ on the right are nonnegative. Thus an upper bound for γ gives a positive terminal gap and a finite terminal scalar flux. Since $\mathrm{d}R=\nu(u,0)\,\mathrm{d}u$, this flux is

$$\int_0^{R_i}\left|\frac{\xi}{\nu}\right|^2\frac{\mathrm{d}R}{R}<\infty.$$

Unitary parallel transport along the terminal cone and Cauchy–Schwarz give (2.9).

The terminal Maxwell equation is

$$\frac{\mathrm{d}Q}{\mathrm{d}R} = q_0 R \,\mathrm{Im}\left(\bar{\phi}\frac{\xi}{\nu}\right).$$

Its right side belongs to $L^1(0, R_i)$, by the scalar flux bound and (2.9). Consequently Q has a finite limit at $R = 0$. This limit is zero: indeed the incoming mass equation reads

$$\frac{\mathrm{d}M}{\mathrm{d}R} = \frac{1-\mu}{2}\left|\frac{\xi}{\nu}\right|^2 + \frac{Q^2}{2R^2} \geq \frac{Q^2}{2R^2},$$

whereas $0 \leq M(R) < R/2$. A nonzero limit of Q would make the mass difference on every fixed terminal interval diverge. Integrating Maxwell from zero and applying Cauchy–Schwarz now gives

$$\begin{aligned}|Q(R)|^2 &\leq q_0^2 \left(\int_0^R \left|\frac{\xi}{\nu}\right|^2 \frac{\mathrm{d}s}{s}\right)\left(\int_0^R s^3|\phi(s)|^2 \,\mathrm{d}s\right)\\ &\leq C q_0^2 R^4 \left(1 + \log(R_i/R)\right)\int_0^R \left|\frac{\xi}{\nu}\right|^2 \frac{\mathrm{d}s}{s}.\end{aligned}$$

This proves (2.10). Since the terminal gap is positive, it also proves

$$\int_0^{R_i} \frac{Q(R)^2}{(1-\mu(R))R^3}\,\mathrm{d}R < \infty. \tag{2.12}$$

The positive terminal integral is the signed integral plus the nonnegative charge correction. By (2.12) its partial integrals are uniformly bounded, so (2.8) follows by monotone convergence. Its converse is immediate. The positive condition gives $\int M(R)R^{-2}\,\mathrm{d}R < \infty$. Since M is nondecreasing in R, integration on $[R, 2R]$ gives $M(R)/R \to 0$. Both the positive integral and the charge correction have finite limits, hence so does their difference γ. Finally $\lambda(u, 0) = \frac{1}{2}\exp(-\gamma(u))$ gives $\lambda_0 > 0$. □

2.2.2 A scalar amplitude bound throughout the original core

The following estimate requires neither the interior gap nor a lower bound for λ.

Lemma 2.4 (Small dimensionless scalar amplitude)**.** *Under the terminal hypothesis,*

$$\sup_{u\leq v\leq 0} r(u, v)|\phi(u, v)| \longrightarrow 0 \qquad (u \uparrow 0).$$

In particular $\sup_{u\leq v\leq 0} |q_0|r|\phi| \to 0$ *for fixed coupling.*

Proof. Let C bound $\lambda/(1-\mu)$ globally. In the gauge $A_v = 0$, outgoing Cauchy–Schwarz and the positive scalar part of the mass equation give

$$\begin{aligned}|\phi(u, v) - \phi(u, 0)|^2 &\leq \left(\int_v^0 \frac{(1-\mu)|\theta|^2}{\lambda}\,\mathrm{d}s\right)\left(\int_v^0 \frac{\lambda}{(1-\mu)r^2}\,\mathrm{d}s\right)\\ &\leq 2CM(R)\frac{-v}{r(u, v)^2}.\end{aligned}$$

Thus

$$\sup_{u\leq v\leq 0} r|\phi| \leq R|\phi(u, 0)| + \sqrt{2CM(R)(-u)} \longrightarrow 0.$$

The terminal logarithmic scalar bound controls the first term; the second tends to zero because $M(R) \leq R/2$ and $u \to 0$. The statement is gauge invariant: in a general gauge the difference in the first line is replaced by parallel-transported scalar values. □

2.2.3 Charge control in an exterior without a core gap

Proposition 2.5 (Adaptive exterior)**.** *There is a positive nondecreasing Lipschitz function ℓ with arbitrarily small Lipschitz constant such that*

$$\ell/R \to 0, \qquad M/\ell \to 0, \qquad \int_0^{R_0} M(R)\ell(R)^{-2}\,\mathrm{d}R < \infty. \tag{2.13}$$

The exterior $\ell(R(u)) \le r \le R(u)$ is invariant under incoming pasts to a fixed proper outgoing row. Uniformly in this exterior,

$$\lambda \longrightarrow \lambda_0 > 0, \qquad \frac{\nu(u,v)}{\nu(u,0)} \longrightarrow 1, \qquad \mathrm{TV}_{[\ell,R]}(\alpha/\lambda) \longrightarrow 0, \qquad \sup_{[\ell,R]} |\theta/\lambda| \longrightarrow 0.$$

The inner frontier is timelike and its normalized scalar datum has finite covariant total variation, with vanishing terminal tail. The same vanishing-tail estimate holds uniformly for any portion of a constant-radius timelike cylinder that starts on a sufficiently late outgoing row and remains in this exterior. In a gauge with $A_v = 0$ and vanishing tangential potential on the inner frontier, ϕ and α/λ converge uniformly throughout the exterior to a common finite complex number. No translation of the scalar is involved.

Proof. Choose $0 < k < 1/8$ and $0 < \sigma_0 < k/2$, and put $\sigma_j = 2^{-j}\sigma_0$. Choose $R_j \downarrow 0$, with $R_{j+1} \le R_j/2$, so that

$$\int_0^{R_j} \frac{M(s)}{s^2}\,\mathrm{d}s \le 2^{-4j}, \qquad \sup_{0<s\le R_j} \frac{M(s)}{s} \le \sigma_j^2.$$

Set $\ell_{\mathrm{step}}(R) = \sigma_j R$ on $(R_{j+1}, R_j]$ and

$$\ell(R) = \sup_{R\le s\le R_0} \{\ell_{\mathrm{step}}(s) - k(s-R)\}.$$

The step function is nondecreasing: within one interval this is immediate, and its upward jumps as R increases also have the required sign. Moreover $\ell \ge \ell_{\mathrm{step}} > 0$. For $R < S$, candidates with $s \ge S$ show $\ell(S) \le \ell(R) + k(S-R)$. Conversely, a candidate at R with $s \ge S$ has value at most its value at S, while a candidate with $R \le s < S$ has value at most $\ell_{\mathrm{step}}(S) \le \ell(S)$. Taking the supremum gives $\ell(R) \le \ell(S)$. Thus $0 \le \ell(S) - \ell(R) \le k(S-R)$.

Since $\ell_{\mathrm{step}}(s) \le (k/2)s$, a positive candidate must have $s < 2R$. Consequently, for $R < R_0/2$,

$$\frac{\ell(R)}{R} \le 2 \sup_{0<s\le 2R} \frac{\ell_{\mathrm{step}}(s)}{s} \longrightarrow 0.$$

For $R \in (R_{j+1}, R_j]$ we also have $M(R)/\ell(R) \le \sigma_j \to 0$, and

$$\int_0^{R_0} \frac{M(R)}{\ell(R)^2}\,\mathrm{d}R \le \sum_{j\ge 0} \sigma_j^{-2} \int_0^{R_j} \frac{M(s)}{s^2}\,\mathrm{d}s \le \sigma_0^{-2} \sum_{j\ge 0} 2^{-2j} < \infty.$$

For sufficiently small R, the exterior $\ell \le r \le R$ has $\mu \le 2M/\ell \le \varepsilon < 1/2$. Its radial scalar energy and scalar variation obey

$$\int_\ell^R \left|\frac{\theta}{\lambda}\right|^2 \mathrm{d}r \le CM, \qquad \mathrm{TV}_{[\ell,R]}\,\phi \le C\sqrt{M/\ell}. \tag{2.14}$$

In particular the terminal logarithmic bound gives, throughout this exterior,

$$|\phi(u,r)| \le L(R), \qquad L(R) = C\left(1+\sqrt{\log(R_i/R)}\right).$$

The notation L is used only as a scalar size bound in this proof.

Uniformly on the adaptive exterior,

$$\frac{Q(u,r)^2}{r^2} \le C\frac{M(R)}{r}\left(1+q_0^2L(R)^2r^2\right) \le C\frac{M(R)}{r} \tag{2.15}$$

after decreasing R_0. To prove this pointwise charge estimate: for $\ell \le r \le R/2$, use the annulus $[r,2r]$; for $R/2 \le r \le R$, use $[R/2,R]$. In both cases the radius on the annulus is comparable to r. The positive electromagnetic part of the radial mass equation gives

$$\int \frac{Q(s)^2}{s^2}\,\mathrm{d}s \le 2M.$$

There is therefore a point on the annulus where $|Q| \le C\sqrt{Mr}$. Maxwell and (2.14) give, between any two points of this annulus,

$$|Q(s)-Q(r)| \le |q_0|L(R)\int s\left|\frac{\theta}{\lambda}\right|\,\mathrm{d}s \le C|q_0|L(R)r^{3/2}\sqrt{M}.$$

Combining the two estimates proves the first inequality. The second follows from $R^2L(R)^2 \to 0$.

This argument does not integrate Maxwell through the uncontrolled core. Thus it uses no interior scalar amplitude bound. From (2.15),

$$\left|\frac{\mu - Q^2/r^2}{1-\mu}\right| \le C\frac{M}{r}.$$

The radial equation for ν now yields

$$\left|\log\frac{\nu(u,r)}{\nu(u,0)}\right| \le C\frac{M}{\ell} \longrightarrow 0. \tag{2.16}$$

For a sufficiently late exterior this ratio exceeds $1/2 > k$. Along an incoming ray, $\mathrm{d}r/\,\mathrm{d}R = \nu(u,v)/\nu(u,0)$; therefore $r-\ell(R)$ strictly increases towards the past while the ray is in the exterior. This proves incoming-past invariance. Moreover, the absolute incoming coefficient is bounded by

$$C(-\nu(u,0))\frac{M(R)}{\ell(R)^2}.$$

The minimum of λ on the exterior portion of a fixed proper row is positive. Integrating $\partial_u \log\lambda$ from that row and using (2.13) therefore gives a positive exterior lower bound for λ. To obtain the uniform limit, split the incoming integrals for $\log\lambda(u,v)$ and $\log\lambda(u,0)$ at a later fixed proper row u_*. Their tails have absolute size at most $C\int_0^{R(u_*)} M\ell^{-2}\,\mathrm{d}R$, uniformly in the incoming label. Their initial difference tends to zero as $u \uparrow 0$ because $u \le v \le 0$ and the fields on $u = u_*$ are continuous. First let $u \uparrow 0$, then $u_* \uparrow 0$. This proves

$$\lambda \to \lambda_0 > 0, \qquad \nu/\nu(u,0) \to 1$$

uniformly throughout the exterior. The frontier label satisfies

$$\lambda(u,v_\ell(u))v_\ell'(u) = (-\nu(u,v_\ell(u))) - \ell'(R)(-\nu(u,0)) \ge (1/2-k)(-\nu(u,0)) > 0 \tag{2.17}$$

almost everywhere. The frontier label $v_\ell(u)$ tends to zero, and the map $u \mapsto v_\ell(u)$ is locally bi-Lipschitz on every proper frontier segment.

The charged AC variation estimate. All quantities here are restricted to one late outgoing row, in the gauge $A_v = 0$. Take the row late enough that $2\ell \le R$ and $1-\mu \ge 1/2$ on $[\ell, R]$. The radial mass equation gives

$$\int_\ell^R \left|\frac{\theta}{\lambda}\right|^2 \,\mathrm{d}r \le 4\{m(u,R) - m(u,\ell)\} \le 4M.$$

There is therefore $r_* \in [\ell, 2\ell]$ with $|\theta/\lambda|(r_*) \le 2\sqrt{M/\ell}$. Also $\partial_r \phi = (\theta/\lambda)/r$, so Cauchy–Schwarz gives

$$\mathrm{TV}_{[\ell,R]}\,\phi \le \left(\int_\ell^R |\theta/\lambda|^2 \,\mathrm{d}r\right)^{1/2} \left(\int_\ell^R r^{-2}\,\mathrm{d}r\right)^{1/2} \le 2\sqrt{M/\ell}.$$

Using $\theta/\lambda = \alpha/\lambda - \phi$ and comparing any point with r_* proves

$$\sup_{[\ell,R]} |\theta/\lambda| \le \mathrm{TV}_{[\ell,R]}(\alpha/\lambda) + 4\sqrt{M/\ell}. \tag{2.18}$$

This uses only the complex triangle inequality and the positive mass flux, and needs no scalar translation.

The exact equation for the rough outgoing derivative is

$$\begin{aligned} D_u\partial_v\left(\frac{\alpha}{\lambda}\right) &= \frac{-\nu}{r}\frac{\mu - Q^2/r^2}{1-\mu}\partial_v\left(\frac{\alpha}{\lambda}\right) \\ &\quad + \frac{\lambda(-\nu)}{(1-\mu)r^2}\left[\left(1-\frac{Q^2}{r^2}\right)\left|\frac{\theta}{\lambda}\right|^2 - 3\mu + \frac{5Q^2}{r^2} + 2q_0Q\,\mathrm{Im}\left(\bar\phi\frac{\theta}{\lambda}\right)\right]\frac{\theta}{\lambda} \\ &\quad + \frac{iq_0\lambda(-\nu)}{(1-\mu)r^2}\left[3Q\frac{\theta}{\lambda} + Q\phi\left(1+\left|\frac{\theta}{\lambda}\right|^2\right) - q_0r^2\phi\,\mathrm{Im}\left(\bar\phi\frac{\theta}{\lambda}\right)\right]. \end{aligned} \tag{2.19}$$

For verification, apply ∂_v to

$$D_u(\alpha/\lambda) = \frac{-\nu}{r}\left[\frac{\mu - Q^2/r^2}{1-\mu}\frac{\theta}{\lambda} + \frac{iq_0Q\phi}{1-\mu}\right]$$

and use $\partial_r = \lambda^{-1}\partial_v$ and

$$\begin{aligned} \partial_r \log\frac{-\nu}{(1-\mu)r} &= \frac{1}{r}\left(\left|\frac{\theta}{\lambda}\right|^2 - 1\right), \\ \partial_r\left(\mu - \frac{Q^2}{r^2}\right) &= \frac{1}{r}\left[(1-\mu)\left|\frac{\theta}{\lambda}\right|^2 - \mu + \frac{3Q^2}{r^2} + 2q_0Q\,\mathrm{Im}\left(\bar\phi\frac{\theta}{\lambda}\right)\right], \\ [D_u, \partial_v]f &= \frac{2iq_0(-\nu)\lambda Q}{(1-\mu)r^2}f. \end{aligned}$$

The electromagnetic commutator is included in the displayed equation. In particular the coefficients of $Q\phi$ and $Q\theta/\lambda$ cannot be obtained by simply replacing μ in the neutral equation.

Using $\lambda\,\mathrm{d}v = \mathrm{d}r$, the exterior optical comparison, (2.14), and (2.15), the absolute integral of the source in (2.19) is bounded by $C(-\nu(u,0))$ times

$$\begin{aligned} &\frac{M}{\ell^2}\left(\sup_{[\ell,R]}|\theta/\lambda| + \sqrt{M/\ell}\right) + |q_0|L\frac{M^{3/2}}{\ell^{3/2}} \\ &\qquad + |q_0|\frac{M}{\ell} + |q_0|L\sqrt{M/\ell} + q_0^2L^2\sqrt{MR}. \end{aligned} \tag{2.20}$$

For clarity, the new terms are obtained from

$$\int_\ell^R \frac{|q_0 Q\phi|}{r^2}\left|\frac{\theta}{\lambda}\right|^2 \,\mathrm{d}r \le C|q_0|LM^{3/2}\ell^{-3/2},$$
$$\int_\ell^R \frac{|q_0 Q|}{r^2}\left|\frac{\theta}{\lambda}\right| \,\mathrm{d}r \le C|q_0|M\ell^{-1},$$
$$\int_\ell^R \frac{|q_0 Q\phi|}{r^2} \,\mathrm{d}r \le C|q_0|L\sqrt{M/\ell},$$
$$\int_\ell^R q_0^2|\phi|^2\left|\frac{\theta}{\lambda}\right| \,\mathrm{d}r \le Cq_0^2L^2\sqrt{MR}.$$

Terms with Q^2/r^2 have the same estimates as the corresponding mass terms. Every term in (2.20) other than $(M/\ell^2)\sup_{[\ell,R]}|\theta/\lambda|$ is integrable in $\mathrm{d}R$. Indeed M/ℓ is bounded, $L(R) = O(\sqrt{1+|\log R|})$, and $\int M\ell^{-2}\,\mathrm{d}R < \infty$. For the second term one can also write

$$|q_0|L\frac{M^{3/2}}{\ell^{3/2}} = \frac{M}{\ell^2}(|q_0|L\ell)\sqrt{M/\ell}.$$

The covariant transport equation implies the ordinary modulus inequality. Integrating it on the moving outgoing interval gives the same favorable frontier exit term as before. In view of (2.18), Gronwall therefore bounds $\mathrm{TV}_{[\ell,R]}(\alpha/\lambda)$. Splitting the incoming representation at a fixed late proper row then proves $\mathrm{TV}_{[\ell,R]}(\alpha/\lambda) \to 0$: its initial contribution is the ordinary L^1 tail of the AC datum, the unitary electromagnetic phase has modulus one, and every source contribution is an integrable tail. Equation (2.18) gives $\sup_{[\ell,R]}|\theta/\lambda| \to 0$.

The exit term is integrable, with vanishing tail. Along the frontier the covariant chain rule gives

$$D_{\partial_u + v_\ell'\partial_v}(\alpha/\lambda) = \frac{-\nu}{r}\left[\frac{\mu - Q^2/r^2}{1-\mu}\frac{\theta}{\lambda} + \frac{iq_0Q\phi}{1-\mu}\right] + v_\ell'\partial_v(\alpha/\lambda).$$

The absolute integral of the first term is bounded by integrable tails of

$$C\frac{M}{\ell^2}\sup_{[\ell,R]}|\theta/\lambda| + C|q_0|L\sqrt{M/\ell}.$$

Thus the frontier normalized datum has finite covariant variation and its terminal variation tail vanishes.

Here are explicit details of the moving-interval argument. Substituting the displayed bound for $\sup_{[\ell,R]}|\theta/\lambda|$ into (2.20), the integrated modulus inequality is

$$\begin{aligned}
&\mathrm{TV}_{[\ell,R]}(\alpha/\lambda)(u) + \int_{u_0}^u v_\ell'(s)\,|\partial_v(\alpha/\lambda)(s, v_\ell(s))|\,\mathrm{d}s \\
&\quad\le \mathrm{TV}_{[\ell,R]}(\alpha/\lambda)(u_0) + C\int_{R(u)}^{R(u_0)} \frac{M}{\ell^2}\,\mathrm{TV}_{[\ell,R]}(\alpha/\lambda)\,\mathrm{d}R \\
&\qquad + C\int_{R(u)}^{R(u_0)}\left[\frac{M}{\ell^2}\sqrt{M/\ell} + |q_0|L\frac{M^{3/2}}{\ell^{3/2}} + |q_0|\frac{M}{\ell} + |q_0|L\sqrt{M/\ell} + q_0^2L^2\sqrt{MR}\right]\mathrm{d}R.
\end{aligned}$$

In particular, after $\mathrm{TV}_{[\ell,R]}(\alpha/\lambda) \to 0$ has been proved, the frontier exit integral tends to zero as $u_0 \uparrow 0$. For the decay assertion, the incoming representation on $[v_\ell(u), 0]$, split at a proper row u_*, has initial contribution bounded by

$$C\int_{v_\ell(u)}^0 |\partial_v(\alpha/\lambda)(u_*, v)|\,\mathrm{d}v \to 0.$$

Its forcing contribution is bounded by the corresponding integrable R-tail in the preceding display. First taking $u \uparrow 0$ and then $u_* \uparrow 0$ gives $\mathrm{TV}_{[\ell,R]}(\alpha/\lambda) \to 0$ without any uniform continuity modulus for the initial derivative.

The same exit computation applies to a fixed cylinder $r = b$ on a time interval $[u_0, u_1]$ on which it remains in the adaptive exterior: its label satisfies $v_b' = (-\nu_b)/\lambda_b > 0$. Use the intervals $[v_b(u), 0]$; their variations are bounded by $\mathrm{TV}_{[\ell,R]}(\alpha/\lambda)(u)$, their forcing by the same exterior integrals, and their initial variation by $\mathrm{TV}_{[\ell,R]}(\alpha/\lambda)(u_0)$. Consequently

$$\int_{u_0}^{u_1} v_b' |\partial_v(\alpha/\lambda)(u, v_b(u))| \, \mathrm{d}u \to 0 \qquad (u_0 \uparrow 0)$$

uniformly in both b and the admissible final time u_1. The covariant chain rule and the already estimated incoming transport term give the identical uniform conclusion for the covariant total variation of $(\alpha/\lambda)|_{r=b}$. This is the precise boundary-variation input needed by the observation argument.

The representative, chain rule, and exit formula used above are those of Lemmas 2.1 and 2.2. Their proofs apply on every proper positive-radius truncation. The terminal estimates then pass by monotone convergence of the nonnegative variation and exit integrals.

One may use the residual gauge depending only on u to set $A_u(u, v_\ell(u)) = 0$. This preserves $A_v = 0$ and all outgoing variations, and changes the AC initial datum only by one constant unit phase. In this gauge the frontier variation is ordinary variation, so α/λ has a finite complex limit there. Since $\sup_{[\ell,R]} |\theta/\lambda| \to 0$ and the radial scalar variation vanishes, ϕ and α/λ converge uniformly throughout the adaptive exterior to the same finite complex constant. This constant is retained; no scalar translation has been made. □

Corollary 2.6 (Small initial interaction norm)**.** *Use the interaction variable*

$$j = (1-\mu)\lambda \left[\alpha' - \frac{1}{r} \left| \frac{\theta}{\lambda} \right|^2 \frac{\theta}{\lambda} \right], \qquad \alpha' = \lambda^{-1} \partial_v (\alpha/\lambda).$$

On the exterior portion of a late outgoing row,

$$\int_{v_\ell(u)}^{0} \left| \frac{\lambda j}{1-\mu} \right| \mathrm{d}v \le \Lambda \, \mathrm{TV}_{[\ell,R]}(\alpha/\lambda) + C\Lambda \sup_{[\ell,R]} |\theta/\lambda| \frac{M}{\ell} \longrightarrow 0, \tag{2.21}$$

where Λ is any uniform upper bound for λ.

Proof. The derivative part is bounded by $\Lambda \int |\partial_v(\alpha/\lambda)| \, \mathrm{d}v$. In the cubic part change variables to $\mathrm{d}r = \lambda \, \mathrm{d}v$ and use

$$\int_\ell^R \left| \frac{\theta}{\lambda} \right|^3 \frac{\mathrm{d}r}{r} \le \sup_{[\ell,R]} |\theta/\lambda| \, \ell^{-1} \int_\ell^R |\theta/\lambda|^2 \, \mathrm{d}r \le C \sup_{[\ell,R]} |\theta/\lambda| \frac{M}{\ell}.$$

No pointwise or squared estimate for the AC derivative is used. □

Scope of these results. The signed terminal condition, the adaptive exterior, the vanishing scalar variations, and $\sup r|\phi| \to 0$ are proved without an interior gap or a uniform bound on incoming blue-shift factors. They do not by themselves give a charged gap in the core. A perturbative charged version of the future-axis annular argument remains the separate geometric input needed for the interior observation and interaction estimates.

2.3 The charged geometric gap at sufficiently small scales

The neutral annular argument is stable under the electromagnetic coupling, but its stability must be established on a region that contains the future regular-axis endpoint of every incoming ray. Stopping at the first outgoing time when a gap fails would not retain this property. The correct continuation parameter is the incoming label of the apex of a proper characteristic triangle.

We use the ordinary Hawking mass m, rather than the renormalized mass $m + Q^2/(2r)$, and set $\mu = 2m/r$. In the gauge $A_v = 0$, with $D_u = \partial_u + iq_0A_u$, the equations needed here are

$$\partial_u\lambda = \partial_v\nu = \frac{\nu\lambda}{r}\frac{\mu - Q^2/r^2}{1-\mu}, \tag{2.22}$$

$$\partial_u m = \frac{1-\mu}{2\nu}|\xi|^2 + \frac{Q^2\nu}{2r^2}, \qquad \partial_v m = \frac{1-\mu}{2\lambda}|\theta|^2 + \frac{Q^2\lambda}{2r^2}, \tag{2.23}$$

$$\partial_v Q = -q_0 r\,\mathrm{Im}(\overline{\phi}\theta), \qquad \partial_v\xi = -\frac{\nu\theta}{r} + \frac{iq_0Q\phi\nu\lambda}{(1-\mu)r}. \tag{2.24}$$

In particular, $m_u \le 0$ and $m_v \ge 0$. All inequalities below are valid for complex scalar fields: the quadratic inequalities are the corresponding inequalities in the real Hilbert space $\mathbb{C}$. The calculations require only the characteristic absolute continuity, the usual local positive-radius bounds, and continuous regular-axis traces; they may equivalently be justified first on proper positive-radius subsets and then by monotone limits.

Lemma 2.7 (Perturbative charged gap)**.** *Let a solution be defined on a proper triangle* $\{u_0 \le u \le v \le b\}$ *with a regular axis,* $\lambda > 0$*,* $\nu < 0$*, and* $0 \le \mu < 1$*. Suppose that, for some* $0 < \delta \le 1$*,*

$$1 - \mu \ge \delta, \qquad |q_0|r|\phi| \le \frac{\delta}{64} \tag{2.25}$$

throughout this triangle. Then

$$1 - \mu \ge \delta_0, \qquad \delta_0 = \frac{\exp(-32)}{256}. \tag{2.26}$$

Proof. Write $\varepsilon = 1/1024$. Radial differentiation on an outgoing row means $\partial_r = \lambda^{-1}\partial_v$. Gauss' law, the zero charge at the center, and Cauchy–Schwarz give

$$Q^2 \le \left(\int_0^r (1-\mu)\left|\frac{\theta}{\lambda}\right|^2 \mathrm{d}s\right)\left(\int_0^r \frac{q_0^2 s^2|\phi|^2}{1-\mu}\,\mathrm{d}s\right) \le 2m\frac{\delta r}{4096}. \tag{2.27}$$

Here the first integral is bounded by $2m$ because the electric contribution to m_r is nonnegative. Consequently

$$\frac{Q^2}{r^2} \le \frac{\delta}{4096}\mu \le \varepsilon\mu. \tag{2.28}$$

The radial identities are

$$\partial_r \log(-\nu) = \frac{\mu - Q^2/r^2}{(1-\mu)r} \ge 0, \tag{2.29}$$

$$\partial_r m = \frac{1-\mu}{2}\left|\frac{\theta}{\lambda}\right|^2 + \frac{Q^2}{2r^2}, \tag{2.30}$$

$$\partial_r\frac{1-\mu}{-\nu} = -\frac{1-\mu}{(-\nu)r}\left|\frac{\theta}{\lambda}\right|^2, \tag{2.31}$$

$$\partial_r \log\frac{\nu^2}{1-\mu} = \frac{1}{r}\left[\frac{\mu - Q^2/r^2}{1-\mu} + \left|\frac{\theta}{\lambda}\right|^2\right] \ge 0. \tag{2.32}$$

The perturbed annular mass estimate. For two incoming rays at an outgoing row, write $r_1 < r_2$, $D = r_2 - r_1$, and

$$\Delta = m_2 - m_1, \qquad \Delta_{\mathrm{sc}} = \frac{1}{2}\int_{r_1}^{r_2} (1-\mu)\left|\frac{\theta}{\lambda}\right|^2 \,\mathrm{d}r.$$

Thus $0 \le \Delta_{\mathrm{sc}} \le \Delta$. Integration of (2.31), using the monotonicity of $(-\nu)r$, gives

$$\frac{(-\nu_2)(1-\mu_1)}{(-\nu_1)(1-\mu_2)} \ge 1 + \frac{2\Delta_{\mathrm{sc}}}{(1-\mu_2)r_2}. \tag{2.33}$$

The scalar equation has the decomposition

$$\xi_2 = \xi_1 + \int_{r_1}^{r_2} \frac{-\nu}{r}\frac{\theta}{\lambda}\,\mathrm{d}r + E, \qquad E = -i\int_{r_1}^{r_2} \frac{q_0 Q\phi(-\nu)}{(1-\mu)r}\,\mathrm{d}r. \tag{2.34}$$

The hypotheses and (2.28) imply

$$|E| \le \frac{\sqrt{\varepsilon}}{64}(-\nu_2)\log\frac{r_2}{r_1}. \tag{2.35}$$

For the scalar integral in (2.34), (2.32) and Cauchy–Schwarz give

$$\left|\int_{r_1}^{r_2} \frac{-\nu}{r}\frac{\theta}{\lambda}\,\mathrm{d}r\right|^2 \le \frac{2\Delta_{\mathrm{sc}}\nu_2^2}{1-\mu_2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Set $F_i = -m_u(u, v_i)$ and

$$F_i^{\mathrm{sc}} = \frac{1-\mu_i}{2(-\nu_i)}|\xi_i|^2, \qquad F_i = F_i^{\mathrm{sc}} + \frac{(-\nu_i)Q_i^2}{2r_i^2}.$$

If the term E is omitted in (2.34), the quadratic inequality $|z+w|^2 - k|z|^2 \le k|w|^2/(k-1)$, $k > 1$, together with (2.33), gives

$$F_{2,\text{without } E}^{\mathrm{sc}} - F_1^{\mathrm{sc}} \le \frac{(-\nu_2)D}{2r_1 r_2}[2\Delta_{\mathrm{sc}} + (1-\mu_2)r_2] \le \frac{(-\nu_2)D}{2r_1}.$$

The last inequality follows from $2\Delta_{\mathrm{sc}} + (1-\mu_2)r_2 \le r_2 - 2m_1 \le r_2$. When $\Delta_{\mathrm{sc}} = 0$, the scalar integral vanishes and $(1-\mu)/(-\nu)$ is constant, so the same conclusion holds without division by Δ_{sc}.

Apply $|z+E|^2 \le (1+\varepsilon)|z|^2 + (1+\varepsilon^{-1})|E|^2$. On annuli with $D/r_1 \le 1$, $\log^2(r_2/r_1) \le D/r_1$, and $(1+\varepsilon^{-1})/4096 < 1$. Therefore

$$F_2 - F_1 \le \varepsilon F_1 + (1+2\varepsilon)\frac{(-\nu_2)D}{2r_1} + \frac{\varepsilon}{2}(-\nu_2). \tag{2.36}$$

This estimate uses no lower bound for the annular scalar mass. In particular, the electromagnetic error has not been divided by a possibly vanishing Δ_{sc}.

Initially let $r_2 = R$, $r_1 = (1-d)R$, $0 < d < 1/2$, and follow the two fixed incoming rays. By (2.29), $D' \le 0$, so $D \le dR$. The outer radius reaches $2dR$ before the inner ray reaches its future regular-axis endpoint. Until this crossing, $r_1 \ge r_2 - dR \ge r_2/2$ and the preceding estimates apply. Furthermore

$$\int F_1\,\mathrm{d}u \le m_{1,\mathrm{initial}} \le \frac{R}{2}, \qquad \int(-\nu_2)\,\mathrm{d}u \le R.$$

Integration of (2.36), using $\Delta' = F_1 - F_2$ and changing variable from u to r_2, gives

$$\frac{2\Delta_{\text{initial}}}{R} < 2d + 2\varepsilon + (1+2\varepsilon)d \log \frac{1-d}{d}. \tag{2.37}$$

At the final radius $2dR$ we used $2\Delta < 2dR$.

The collar of a near-trapped sphere. Fix $d = 1/8$. At a point with $1 - \mu_2 \le 1/256$, apply (2.37) with every fractional width $0 < s \le d$. Throughout the inner collar this gives

$$1 - \mu((1-s)r_2) \le \frac{1/256 + 2\varepsilon + s + (1+2\varepsilon)s\log((1-s)/s)}{1-s} < \frac{7}{16}. \tag{2.38}$$

Indeed the displayed rational function is increasing on $(0, 1/8]$. At $s = 1/8$, the elementary bound $\log 7 < 2$ bounds it by $781/1792 < 7/16$. Consequently

$$\frac{\mu - Q^2/r^2}{1-\mu} \ge (1-\varepsilon)\frac{9}{7} > 1 \tag{2.39}$$

on the collar.

Fix an incoming ray whose gap is initially $c < 1/256$, at radius r_*. Follow it until its first gap value $1/256$, which is attained before its future regular-axis endpoint. On this interval let the moving inner boundary be $r_1 = (1-d)r_2$. Equation (2.39) gives

$$\frac{-\nu_1}{-\nu_2} \le 1 - d, \qquad \lambda_1 v_1' = (-\nu_1) - (1-d)(-\nu_2) \le 0.$$

Since $m_v \ge 0$, the inner mass budget is

$$\int F_1 \, \mathrm{d}u \le \frac{(1-d)r_*}{2}. \tag{2.40}$$

The same estimate holds on every initial subinterval.

The mass-flux comparison on the moving collar. From (2.39) and (2.28),

$$\frac{\nu^2}{(1-\mu)r^2} \le \frac{2}{1-\varepsilon}\partial_r\left(\frac{\nu^2}{r}\right).$$

To check this inequality, set temporarily $t = (\mu - Q^2/r^2)/(1-\mu) \ge 1$; then $1/(1-\mu) \le 1 + t/(1-\varepsilon) \le 2(2t-1)/(1-\varepsilon)$. Thus

$$\left|\int_{r_1}^{r_2} \frac{-\nu}{r}\frac{\theta}{\lambda}\,\mathrm{d}r\right|^2 \le \frac{4\Delta_{\text{sc}}\nu_2^2}{(1-\varepsilon)r_2}. \tag{2.41}$$

Moreover, the electric part of the annular mass satisfies $2(\Delta - \Delta_{\text{sc}}) = \int_{r_1}^{r_2} Q^2/r^2 \, \mathrm{d}r \le \varepsilon d r_2$. It follows from (2.33) and $2m_1 < r_1$ that

$$\frac{(1-\mu_2)(-\nu_1)}{(1-\mu_1)(-\nu_2)} \le \frac{1-\mu_2}{d(1-\varepsilon)}. \tag{2.42}$$

Apply $|z+w+E|^2 \le 2(1+\varepsilon)(|z|^2 + |w|^2) + (1+\varepsilon^{-1})|E|^2$ to (2.34). Equations (2.35), (2.41), and (2.42), together with $2\Delta_{\text{sc}} \le r_2$ and $\log(8/7) < 1$, imply

$$F_2^{\text{sc}} \le \frac{9(1-\mu_2)}{4d}F_1 + \frac{9}{4}(1-\mu_2)(-\nu_2). \tag{2.43}$$

For completeness, the two coefficients before rounding upward are $2(1+\varepsilon)/[d(1-\varepsilon)]$ and at most $2(1+\varepsilon)/(1-\varepsilon)+\varepsilon/2$; both are bounded by the stated coefficients.

Recovery of the gap. Before first exit, $1-\mu_2 \le 1/256$ and $\int(-\nu_2)\,\mathrm{d}u \le r_*$. Adding the electric flux $(-\nu_2)Q_2^2/(2r_2^2) \le \varepsilon(-\nu_2)/2$ to (2.43), and using (2.40), gives

$$\int F_2\,\mathrm{d}u \le \left[\frac{9(1-d)}{8d\cdot 256}+\frac{9}{4\cdot 256}+\frac{\varepsilon}{2}\right] r_* = \frac{41}{1024}r_*.$$

Thus $m_2 \ge 469r_*/1024$ and $r_2 > r_*/2$ throughout this interval. Crucially, keep the electric flux together with the geometric term in the equation for the gap:

$$\begin{aligned}\frac{d}{du}(1-\mu_2) &= \frac{2F_2^{\mathrm{sc}} - (\mu_2 - Q_2^2/r_2^2)(-\nu_2)}{r_2} \\ &\le \frac{9(1-\mu_2)F_1}{2dr_2} + \frac{[\frac{9}{2}(1-\mu_2)-(1-\varepsilon)\mu_2](-\nu_2)}{r_2} \le \frac{9(1-\mu_2)F_1}{2dr_2}.\end{aligned}$$

The second term is nonpositive when $1-\mu_2 \le 1/256$. Division by the positive gap and integration to first exit yields

$$\log\frac{1}{256c} \le \frac{9}{dr_*}\int F_1\,\mathrm{d}u \le \frac{9(1-d)}{2d} = \frac{63}{2} < 32.$$

Therefore $c > \exp(-32)/256$, proving (2.26). The axis and other points already satisfy the conclusion. □

Proposition 2.8 (Late uniform gap from the terminal estimates)**.** *Suppose the charged development has the preceding local regularity, a regular axis up to the missing endpoint, and $\lambda/(1-\mu) \le \Lambda$. Set $R(u) = r(u,0)$ and $M(u) = m(u,0)$. Assume that the terminal estimates give*

$$R(u)|\phi(u,0)| + \sqrt{2\Lambda M(u)(-u)} \longrightarrow 0 \quad \text{as } u \uparrow 0. \tag{2.44}$$

Then, for all sufficiently late $u_0 < 0$,

$$1-\mu \ge \frac{\delta_0}{2} \quad \text{on } \{u_0 \le u \le v < 0\}. \tag{2.45}$$

By continuity the same holds at positive-radius terminal-cone points.

Proof. No scalar translation is used. On an outgoing row, Cauchy–Schwarz and (2.30) imply

$$\begin{aligned}r^2|\phi(u,v)-\phi(u,0)|^2 &\le \left(\int_r^R (1-\mu)\left|\frac{\theta}{\lambda}\right|^2 \mathrm{d}s\right)\left(\int_r^R \frac{r^2}{(1-\mu)s^2}\,\mathrm{d}s\right) \\ &\le 2M(u)\Lambda(-v) \le 2\Lambda M(u)(-u).\end{aligned}$$

The second line uses $\mathrm{d}s = \lambda\,\mathrm{d}v$ and $\lambda/(1-\mu) \le \Lambda$. Consequently, throughout the outgoing row,

$$r|\phi(u,v)| \le R(u)|\phi(u,0)| + \sqrt{2\Lambda M(u)(-u)}. \tag{2.46}$$

Choose u_0 so late that $|q_0|r|\phi| \le \delta_0/128$ for every $u \ge u_0$. For $u_0 \le b < 0$, consider the nested proper triangles

$$\mathcal{T}_b = \{u_0 \le u \le v \le b\}, \qquad H(b) = \min_{\mathcal{T}_b}(1-\mu).$$

Local continuity, including the regular-axis trace, implies that H is continuous and $H(u_0) = 1$. If it ever reached $\delta_0/2$, let b_* be the first such value. On $\mathcal{T}_{b_*}$ all hypotheses of Lemma 2.7 hold with $\delta = \delta_0/2$, and every incoming ray has its future regular-axis endpoint inside the same proper triangle. The lemma gives $H(b_*) \ge \delta_0$, a contradiction. This proves (2.45). □

Remark 2.9 (Dependence on the terminal hypothesis). The conditional geometric lemma does not assume a bound on the terminal blue-shift factor. Its smallness assumption is obtained only at late scales from (2.44). In the intended application, the positive terminal integral $\int(-\nu)\mu/[(1-\mu)r]\,\mathrm{d}u < \infty$ gives the terminal estimates $R \to 0$, $M \le R/2$, and $|\phi(u,0)| = O(1+\sqrt{\log(R_i/R(u))})$, which imply (2.44). This separates the geometric bootstrap from the terminal estimates and avoids assuming a uniform bound on the interior blue-shift factors at any stage.

2.4 The charged cylinder identity and observation

We use the ordinary Hawking mass m, put $\mu = 2m/r$, and use the convention $D = \mathrm{d} + iq_0 A$. Thus the equations relevant here are

$$\begin{aligned} r_{uv} &= \frac{\nu\lambda}{r}\frac{\mu - Q^2/r^2}{1-\mu}, \\ m_v &= \frac{(1-\mu)|\theta|^2}{2\lambda} + \frac{Q^2\lambda}{2r^2}, & m_u &= \frac{(1-\mu)|\xi|^2}{2\nu} + \frac{Q^2\nu}{2r^2}, \\ (Q)_v &= -q_0 r\,\mathrm{Im}(\bar\phi\theta), & D_u\theta &= -\frac{\lambda}{r}\xi - \frac{iq_0 Q\nu\lambda}{(1-\mu)r}\phi. \end{aligned} \tag{2.47}$$

In the gauge $A_v = 0$,

$$(A_u)_v = \frac{2\nu\lambda Q}{(1-\mu)r^2}.$$

We reserve Q_b for the cylinder energy and write Q_{cyl} for the restriction of the electric charge Q to that cylinder. The regular-axis conditions are $m = Q = 0$. All absolute values below are complex moduli.

2.4.1 There is no electric error in the cylinder identity

On $r = b$ use the clock

$$\mathrm{d}\tau = (-\nu_b)\,\mathrm{d}u = \lambda_b\,\mathrm{d}v_b, \qquad D_\tau = (-\nu_b)^{-1}D_u + \lambda_b^{-1}D_v.$$

The two exact identities

$$bD_\tau\phi = \left(\frac{\theta}{\lambda} - \frac{\xi}{\nu}\right)_b, \qquad b\partial_\tau\mu_b = (1-\mu_b)\left(\left|\frac{\theta}{\lambda}\right|_b^2 - \left|\frac{\xi}{\nu}\right|_b^2\right) \tag{2.48}$$

follow directly from (2.47). In the second identity the two electric contributions cancel, because the derivative is tangent to a constant-radius cylinder. Set

$$Q_b = \frac{b}{2}\left\{\left|\frac{\theta}{\lambda}\right|_b^2 - \log(1-\mu_b)\right\}.$$

Since $\alpha/\lambda = \phi + \theta/\lambda$, differentiation gives

$$\frac{\mathrm{d}Q_b}{\mathrm{d}\tau} + \frac{1}{2}\left|\frac{\theta}{\lambda} - \frac{\xi}{\nu}\right|_b^2 = b\,\mathrm{Re}\left[\overline{\left(\frac{\theta}{\lambda}\right)_b} D_\tau(\alpha/\lambda)_b\right]. \tag{2.49}$$

Indeed, subtracting $b\,\mathrm{Re}(\overline{\theta/\lambda}D_\tau\phi)$ from the derivative of the first term of Q_b and using (2.48) leaves $-\frac{1}{2}|\theta/\lambda - \xi/\nu|^2$. The connection contributes no derivative of a modulus. We use the covariant variation

$$\mathrm{TV}_I^D((\alpha/\lambda)_b) := \int_I |D_\tau(\alpha/\lambda)_b|\,\mathrm{d}\tau.$$

2.4.2 A quantitative charged observation statement

The last-cone hypothesis leads to observation of μ_b^2. We compare with a static reference down to a positive rescaled radius, deriving the required lapse bounds there. The core gap is supplied by Proposition 2.8; the adaptive exterior and the later stopping argument supply small boundary θ/λ. The uniformly finite-blueshift observation theorem [13, Theorem 3.3], which uses travel all the way to the axis, is not invoked.

Proposition 2.10 (Charged observation at small physical radius)**.** *Fix $0 < \delta_0 < 1$. There are positive constants $T_*, C_*, \delta_*, \eta_*$ depending only on δ_0 such that the following holds. Consider a cylinder $r = b$ in a regular-center development. Every outgoing section used in its inward comparison region is assumed to join the existing regular axis, and throughout these sections assume*

$$m \geq 0, \qquad 1 - \mu \geq \delta_0.$$

Let $I = [\tau_0 - T_* b, \tau_0 + T_* b]$ and its inward dependence region lie in the original development. Suppose $(\alpha/\lambda)_b$ is covariantly AC and

$$\mu_b + \left|\frac{\theta}{\lambda}\right|_b \leq \delta_* \quad \text{on } I, \qquad |q_0| b\,(1 + |(\alpha/\lambda)_b(\tau_0)|) \leq \eta_*. \tag{2.50}$$

Then

$$b\left\{\left|\frac{\theta}{\lambda}\right|_b^2(\tau_0) + \mu_b(\tau_0)^2\right\} \leq C_*\left\{\int_I \left|\frac{\theta}{\lambda} - \frac{\xi}{\nu}\right|_b^2 \mathrm{d}\tau + b\left[\mathrm{TV}_I^D((\alpha/\lambda)_b)\right]^2\right\}. \tag{2.51}$$

There is no hypothesis on a lower bound for the original-coordinate expansions and no hypothesis on $\sup_{r<b}|\phi|$.

Proof. The reference has an uncharged complex scalar and constant Maxwell charge. Its charge is a compact comparison parameter; scalar translation is not used as a symmetry of the charged equations.

Gauge, normalization, and boundary information. First choose $A_v = 0$. The residual gauge transformation depending only on u can then make $A_u = 0$ on the outer cylinder. Thus the connection vanishes along its tangent, and covariant boundary variation is ordinary variation in this frame. Write

$$H = (\alpha/\lambda)_b(\tau_0)$$

in this frame. If $m_b(\tau_0) + b|\theta/\lambda|_b(\tau_0) = 0$, the conclusion is immediate. Otherwise divide lengths, the Hawking mass, and the electric charge by $m_b(\tau_0) + b|\theta/\lambda|_b(\tau_0) + |Q_{\mathrm{cyl}}(\tau_0)|$, and set

$$L = \frac{b}{m_b(\tau_0) + b|\theta/\lambda|_b(\tau_0) + |Q_{\mathrm{cyl}}(\tau_0)|},$$
$$\widehat{q}_0 = q_0\{m_b(\tau_0) + b|\theta/\lambda|_b(\tau_0) + |Q_{\mathrm{cyl}}(\tau_0)|\}.$$

The scalar field is unchanged. The rescaled equations have the form (2.47), with coupling $\widehat{q}_0$. Put

$$\eta = |\widehat{q}_0| L(1 + |H|) = |q_0| b(1 + |H|). \tag{2.52}$$

Normalize both null labels to $t = 2(\tau - \tau_0)/\{m_b(\tau_0) + b|\theta/\lambda|_b(\tau_0) + |Q_{\mathrm{cyl}}(\tau_0)|\}$ on the outer cylinder. Thus $U = V = t$ there, $U \geq V$ inward, and

$$r = L, \quad \nu = -\tfrac{1}{2}, \quad \lambda = \tfrac{1}{2}, \quad A_V = 0, \quad A_U(t,t) = 0.$$

The axis conditions used below, $m = Q = r\phi = 0$, are gauge invariant; the comparison does not require zero axis potential. Put $T = 2T_*$.

The complete inward region and its characteristic triangles. In the normalized chart put $a = -TL$, $c = TL$. The complete inward region in the proposition is

$$\mathcal{D}[a,c] = \{(U,V) \text{ in the original development} : a \le V \le U \le c, 0 \le r \le L\}.$$

Completeness here means that each incoming segment beginning at (V,V) is present up to $U = c$ or its first intersection with the existing axis, whichever comes first, and that its points have their outgoing segments to (U,U) present. This is part of the meaning of the inward-region hypothesis for a general development. For the domain $\mathcal{Q}$ of Theorem 1.1, it follows directly from $u_i \le u \le v \le 0$ and the monotonicity of the two changes of null label.

For a positive-radius point $p = (U_p, V_p)$ in this region define its sideways triangle by

$$\mathcal{T}_p = \{(U,V) : V_p \le V \le U \le U_p\}.$$

Completeness puts this triangle in the development, and monotonicity gives

$$r(U,V) \ge r(U_p,V_p) > 0.$$

For $(U,V) \in \mathcal{T}_p$, the incoming segment $\{(s,V) : V \le s \le U\}$ and outward outgoing segment $\{(U,s) : V \le s \le U\}$ stay in $\mathcal{T}_p$ at radius at least $r(U,V)$. This verifies all characteristic and outward Maxwell integrations used below, including on radial truncations.

The radial mass and charge constraints also use the assumed full outgoing sections to the axis. Those sections may leave $\mathcal{T}_p$, but use only their geometric gap and regular-axis values there. We use m, r, Q, ν, λ for the rescaled fields. The normalized central parameters are

$$M = m_b(0), \quad P = L(\phi_b(0) - H), \quad Q_* = Q_{\mathrm{cyl}}(0), \quad M + |P| + |Q_*| = 1.$$

Here P is complex and Q_* is real. The added charge enlarges the scaling denominator and therefore only improves the normalized small-error bounds in the contradiction argument.

To bound L from below, first use the radial mass equation and Cauchy–Schwarz in physical variables:

$$|\phi(r)| \le |\phi_b| + (2m_b/\delta_0)^{1/2} r^{-1/2}, \qquad \frac{|Q_{\mathrm{cyl}}|^2}{b^2} \le C_{\delta_0} |q_0|^2 b^2 \mu_b (|\phi_b|^2 + \mu_b).$$

At the central point $\phi_b = H - (\theta/\lambda)_b$. For $\delta_* \le 1$, this yields $|Q_{\mathrm{cyl}}|/b \le C_{\delta_0} \eta \sqrt{\mu_b}$. Therefore

$$L^{-1} = \mu_b(\tau_0)/2 + |\theta/\lambda|_b(\tau_0) + |Q_{\mathrm{cyl}}(\tau_0)|/b \le \delta_* + C_{\delta_0} \eta_* \sqrt{\delta_*}. \tag{2.53}$$

Thus choosing δ_* after η_* ensures $L \ge L_0$.

Failure of (2.51), with arbitrarily large C_*, would give, on $[-TL, TL]$ for a fixed sufficiently large $T = 2T_*$,

$$\sup |\alpha/\lambda - H|_b \le \epsilon/L, \qquad \|\phi_b - H + (\xi/\nu)_b\|_{L^2(\,\mathrm{d}t)} \le \epsilon/\sqrt{L} \tag{2.54}$$

with arbitrarily small ϵ. Constants depending on the chosen T have been included in ϵ. This uses

$$\phi - H + \xi/\nu = (\alpha/\lambda - H) - (\theta/\lambda - \xi/\nu).$$

At the boundary the exact equations are

$$2L\phi_t = (\theta/\lambda - \xi/\nu)_b, \qquad 4m_t = (1 - \mu_b)\{|\theta/\lambda|_b^2 - |\xi/\nu|_b^2\}.$$

The electric terms cancel in the second equation. Integration, Cauchy–Schwarz, and $|z_1|^2 - |z_2|^2 = \mathrm{Re}((\bar z_1 + \bar z_2)(z_1 - z_2))$ give

$$\sup|\phi_b - H - P/L| \le C_T\epsilon/L, \qquad \sup|m_b - M| \le C_T\epsilon/L. \tag{2.55}$$

In particular $m_b \le 2$ for small ϵ. Outgoing mass monotonicity implies $0 \le m \le 2$ on every full outgoing section from the regular axis to this portion of the cylinder. The boundary charge satisfies

$$\partial_t Q_{\mathrm{cyl}} = -\frac{\widehat{q}_0 L}{2}\,\mathrm{Im}\left[\bar\phi_b(\theta/\lambda + \xi/\nu)_b\right].$$

The L^2 norm of the last sum is at most $C_T/\sqrt{L}$, and $|\phi_b| \le |H| + C_T/L$. Cauchy–Schwarz on a window of length $2TL$ therefore gives

$$\sup|Q_{\mathrm{cyl}} - Q_*| \le C_T\eta. \tag{2.56}$$

Control of the core charge without a scalar supremum. For an outgoing section, $\partial_r = \lambda^{-1}\partial_V$, and $A_V = 0$. The radial mass equation gives

$$\int_0^L \left|\frac{\theta}{\lambda}\right|^2 \mathrm{d}r \le \frac{2m(L)}{\delta_0} \le \frac{4}{\delta_0}.$$

Since $\partial_r\phi = (\theta/\lambda)/r$,

$$|\phi(r)| \le |\phi(L)| + C_{\delta_0} r^{-1/2} \le |H| + C_{T,\delta_0} r^{-1/2}, \qquad 0 < r \le L. \tag{2.57}$$

Here $L \ge 1$ and the small $O(L^{-1})$ boundary error is absorbed. Maxwell's equation and $Q(0) = 0$ give

$$\begin{aligned} Q(r)^2 &\le \widehat{q}_0^2 \left(\int_0^r (1-\mu)|\theta/\lambda|^2\,\mathrm{d}s\right)\left(\int_0^r \frac{s^2|\phi|^2}{1-\mu}\,\mathrm{d}s\right) \\ &\le C_{T,\delta_0}\widehat{q}_0^2 m(r)(|H|^2 r^3 + r^2). \end{aligned} \tag{2.58}$$

Consequently

$$\frac{Q^2}{r^2} \le C_{T,\delta_0}\widehat{q}_0^2\mu(|H|^2 r^2 + r) \le C_{T,\delta_0}\eta^2\mu. \tag{2.59}$$

If η is sufficiently small, then

$$0 \le \frac{\mu - Q^2/r^2}{1-\mu} \le \frac{\mu}{1-\mu}. \tag{2.60}$$

In particular the normalized expansions decrease inward, and their lower bounds follow from exactly the neutral radial integrals. On a fixed outer annulus $r \ge L_0 \ge 8$, for example,

$$\tfrac{1}{2}\exp(-8/L_0) \le -\nu, \lambda \le \tfrac{1}{2}. \tag{2.61}$$

For either null segment from the outer cylinder to a point at $r \ge r_* > 0$, the logarithmic expansion equation, $m \le 2$, and the gap give

$$0 \le \int_r^L \frac{\mu - Q^2/s^2}{(1-\mu)s}\,\mathrm{d}s \le \frac{4}{\delta_0}\int_r^L \frac{\mathrm{d}s}{s^2} \le \frac{4}{\delta_0 r_*}. \tag{2.62}$$

Hence $\frac{1}{2}\exp\{-4/(\delta_0 r_*)\} \le -\nu, \lambda \le 1/2$ on those segments. We will fix $r_* > 0$ using the reference barriers; the estimate asserts no uniform travel bound to the axis.

The static family with constant electric charge. Construct a static Einstein–Maxwell reference with an uncharged complex scalar, coupling zero and constant electric charge Q_*. Its boundary data are

$$m_*(L) = M, \quad \phi_*(L) = P/L, \quad (\theta_*/\lambda_*)(L) = -P/L, \quad \lambda_*(L) = \tfrac{1}{2}.$$

Its equations are

$$\begin{aligned} \partial_r \log \lambda_* &= \frac{\mu_* - Q_*^2/r^2}{(1-\mu_*)r}, & \partial_r m_* &= \frac{1-\mu_*}{2}|\theta_*/\lambda_*|^2 + \frac{Q_*^2}{2r^2}, \\ \partial_r(\theta_*/\lambda_*) &= -\frac{1 - Q_*^2/r^2}{(1-\mu_*)r}\frac{\theta_*}{\lambda_*}, & \partial_r \phi_* &= \frac{\theta_*/\lambda_*}{r}, \end{aligned} \tag{2.63}$$

and $2r\theta_* = -P$. The scalar has constant complex phase, so the radial geometry depends only on $|P|$ and Q_*^2. At large fixed L_0, uniformly over $M + |P| + |Q_*| = 1$ and $L \geq L_0$,

$$|m_*| \leq 2, \quad 1 - \mu_* \geq \tfrac{1}{2}, \quad |\phi_*| + |\theta_*/\lambda_*| \leq C/r, \quad |\partial_r(r\phi_*)| \leq C/r^2. \tag{2.64}$$

Existence of the compact static family and its strict barriers. Write $a = L^{-1}$ and $x = r^{-1}$. The parameter set is

$$\{(M, P, Q_*, a) : M \geq 0, \quad M + |P| + |Q_*| = 1, \quad 0 \leq a \leq x_0\}, \qquad x_0 = L_0^{-1}.$$

The conserved relation $2r\theta_* = -P$ converts the radial system to

$$\begin{aligned} \frac{\mathrm{d}m_*}{\mathrm{d}x} &= -\frac{(1-2xm_*)|P|^2}{8\lambda_*^2} - \frac{Q_*^2}{2}, \\ \frac{\mathrm{d}\lambda_*}{\mathrm{d}x} &= -\lambda_*\frac{2m_* - Q_*^2 x}{1-2xm_*}, & \frac{\mathrm{d}\phi_*}{\mathrm{d}x} &= \frac{P}{2\lambda_*}. \end{aligned}$$

Treating complex scalars as pairs of real variables, consider the box

$$0 \leq x \leq x_0, \quad -1 \leq m_* \leq 2, \quad \tfrac{1}{4} \leq \lambda_* \leq \tfrac{3}{4},$$

For $x_0 \leq 1/16$, the gap is at least $3/4$ and the vector field is uniformly bounded and Lipschitz. Starting from $(M, 1/2, aP)$ at $x = a$, choose x_0 times its vector-field bound smaller than the distance of $(M, 1/2)$ to the m_*, λ_* faces. The integral equation and a first-exit argument then give existence on every $[a, x_0]$, including $a = 0$.

The same integral estimates give

$$|m_*(x) - M| + |\lambda_*(x) - \tfrac{1}{2}| \leq Cx, \qquad |\phi_*(x)| \leq Cx.$$

Continuous dependence for this bounded Lipschitz ODE, including its variable starting point, shows that the data at $x = x_0$ form a compact set. The conserved relation $2r\theta_* = -P$ then gives $|\theta_*/\lambda_*| \leq C/r$. Integrating $\partial_r^2(r\phi_*) = -(\mu_* - Q_*^2/r^2)(\theta_*/\lambda_*)/((1-\mu_*)r)$ backwards from $\partial_r(r\phi_*)(L) = 0$ proves the last estimate in (2.64). All these conclusions are uniform in the compact parameter set.

Fix one member. If it already satisfies $m_*(L_0) < 0$ or $1 - \mu_*(L_0) < \delta_0$, with positive gap, then L_0 itself is a strict barrier. Otherwise continue its ODE inward as long as

$$m_* \geq 0, \qquad 1 - \mu_* \geq \delta_0/2.$$

Since $\partial_r m_* \geq 0$, we have $0 \leq m_*(r) \leq 2$ throughout this continuation. On every interval $\rho \leq r \leq L_0$, $\rho > 0$, the logarithmic equation for λ_* gives

$$|\partial_r \log \lambda_*| \leq \frac{2}{\delta_0}\left(\frac{4}{r^2} + \frac{Q_*^2}{r^3}\right).$$

This bounds λ_* above and away from zero; the remaining ODE derivatives are then bounded. ODE continuation excludes a positive-radius endpoint before either stopping condition is reached.

The continuation cannot retain both conditions down to zero. If $Q_* \neq 0$, integration of $\partial_r m_* \geq Q_*^2/(2r^2)$ would give

$$m_*(r) \leq m_*(L_0) - \frac{Q_*^2}{2}\left(\frac{1}{r} - \frac{1}{L_0}\right),$$

contrary to nonnegative mass for sufficiently small r. If $Q_* = 0$ and $P \neq 0$, the logarithmic expansion equation has nonnegative right-hand side, so $\lambda_*(r) \leq \lambda_*(L_0)$ inward. Hence

$$\partial_r m_* = \frac{(1-\mu_*)|P|^2}{8r^2\lambda_*^2} \geq \frac{\delta_0|P|^2}{16\lambda_*(L_0)^2 r^2},$$

which gives the same contradiction. Finally $P = Q_* = 0$ forces $M = 1$; then $m_* \equiv 1$ and $1 - \mu_* = 1 - 2/r$ reaches $\delta_0/2$ at a positive radius.

Reaching $1 - \mu_* = \delta_0/2$ strictly violates the required gap while the ODE remains nondegenerate. At a first zero of m_*, at least one of P, Q_* is nonzero, so $\partial_r m_* > 0$; continuing slightly inward gives $m_* < 0$ with positive gap. Each member therefore reaches a strict positive-radius barrier.

For each member choose such a radius ρ and strict margin. Continuous dependence on $[\rho, L_0]$ preserves positivity of the gap and expansion and half the violation margin in a parameter neighborhood. A finite cover gives common $r_*, \gamma > 0$ and coefficient bounds: every member is defined down to a selected $\rho_p \in [r_*, L_0]$ at which

$$m_*(\rho_p) \leq -\gamma \quad \text{or} \quad 1 - \mu_*(\rho_p) \leq \delta_0 - \gamma,$$

with positive gap and expansion down to that radius. No continuation below its selected barrier is required.

A comparison point and a single complete comparison triangle. Let ρ_p be the barrier radius for the selected reference member. If $\rho_p = L$, the boundary mass and the reference mass coincide at $t = 0$, contradicting the strict reference barrier immediately. Otherwise put

$$c_* = \tfrac{1}{2}\exp\{-4/(\delta_0 r_*)\}, \qquad T > 4/c_*.$$

Follow the incoming ray $V = 0$ from $(0,0)$ on the outer cylinder. As long as its radius is at least ρ_p, the optical estimate (2.62) gives $-\nu \geq c_*$. If the ray meets the existing axis, continuity of r forces it to pass through ρ_p first. If it does not meet the axis before $U = TL/2$, failure to reach ρ_p would give

$$L - r(TL/2, 0) = \int_0^{TL/2} (-\nu)(U, 0)\,\mathrm{d}U \geq c_* TL/2 > L,$$

a contradiction. Completeness of the inward region excludes any other endpoint of this integration. There is therefore a point $p = (U_p, 0)$ with $r(p) = \rho_p$ and $0 < U_p \leq L/c_* < TL/4$.

Fix this $\mathcal{T}_p$ for all subsequent comparisons. It stays in the original development at $r \geq \rho_p$, with boundary labels in $[0, U_p] \subset [-TL, TL]$.

The Maxwell improvement on the outer annulus. Define comparison variables, in the fixed boundary-parallel gauge,

$$\psi = r(\phi - H), \qquad w = \psi - r\phi_*(r), \qquad \delta m = m - m_*(r).$$

These are differences used to compare solutions; $\phi - H$ is not asserted to solve the charged scalar equation. For $L_0 \le \rho \le L$ set

$$\mathcal{T}_p(\rho) = \mathcal{T}_p \cap \{r \ge \rho\}, \qquad N_\phi(\rho) = \max_{\mathcal{T}_p(\rho)} |w|, \qquad N_m(\rho) = \max_{\mathcal{T}_p(\rho)} |\delta m|.$$

For a fixed truncation we abbreviate these maxima by N_ϕ, N_m. The maxima are finite and continuous in ρ. Indeed the triangle is a proper compact positive-radius set. If a point has r between two nearby cutoffs, move it outward at fixed U until the larger radius is reached. This segment remains in $\mathcal{T}_p$ and, since λ has a positive minimum there, its coordinate length is bounded by a constant times the difference of the cutoffs. Thus the truncated sets vary continuously in Hausdorff distance; uniform continuity of w and δm proves continuity of their maxima. At $\rho = L$ the set is a portion of the outer diagonal and $N_\phi(L) \le C_T\epsilon$. Temporarily impose $N_\phi(\rho) \le 1$. Equation (2.64) gives $|\psi| \le C$ on this entire truncation, and the mass bounds give $N_m(\rho) \le 4$ without a further bootstrap assumption.

Uniformity as $L \to \infty$ requires integrating Maxwell's equation by parts from the outer cylinder:

$$Q(r) = Q_{\mathrm{cyl}}(U) + \widehat{q}_0 \operatorname{Im}\left\{\bar{H}\left[L\psi_b(U) - r\psi(r) - 2\int_r^L \psi(s)\,\mathrm{d}s\right] + \int_r^L \bar{\psi}\,\partial_s\psi\,\mathrm{d}s\right\}. \tag{2.65}$$

To check it, use $\theta/\lambda = \partial_r\psi - \psi/r$ and expand $r\operatorname{Im}(\bar{\phi}\,\theta/\lambda)$. Moreover

$$\operatorname{Im}\int \bar{\psi}\partial_s\psi\,\mathrm{d}s = \operatorname{Im}\int \bar{\psi}(\theta/\lambda)\,\mathrm{d}s,$$

because the omitted integrand $|\psi|^2/s$ is real. The mass bound and Cauchy–Schwarz therefore bound this term by $C\sqrt{L}$. The term containing H is at most $C|H|L$. Together with (2.56), this proves

$$|Q(r) - Q_*| \le C_{T,\delta_0}\eta, \qquad |Q(r)| \le 2 \tag{2.66}$$

for small η, uniformly in L. The segment $[r, L]$ stays in the truncated sideways triangle, so the bootstrap bound on ψ applies throughout it.

Integrate Maxwell's equation for A_U at fixed U, starting from $A_U(U, U) = 0$. The upper expansion bound gives

$$|A_U(r)| \le C_{\delta_0}\int_r^L \frac{|Q(s)|}{s^2}\,\mathrm{d}s.$$

Consequently

$$|A_U(r)| \le C_{T,\delta_0}/r, \qquad |\widehat{q}_0 A_U(r)|(1 + |H|) \le C_{T,\delta_0}\frac{\eta}{Lr}. \tag{2.67}$$

Exact comparison equation and uniform error estimates. The shifted scalar satisfies

$$\begin{aligned}\partial_U\partial_V\psi = &-\frac{(-\nu)\lambda}{r^2}\frac{\mu - Q^2/r^2}{1-\mu}\psi - i\widehat{q}_0 A_U(\partial_V\psi + \lambda H)\\ &+ \frac{i\widehat{q}_0(-\nu)\lambda Q}{(1-\mu)r^2}(\psi + rH).\end{aligned} \tag{2.68}$$

In particular, the terms involving H have not been discarded. Subtracting the radial static equation and putting the derivative coupling on the left gives the useful exact identity

$$\begin{aligned} D_U(\partial_V w) = & -\frac{(-\nu)\lambda}{r^2}\frac{\mu - Q^2/r^2}{1-\mu}\, w \\ & + \frac{(-\nu)\lambda}{r}\frac{\theta_*}{\lambda_*}\left\{\frac{\mu - Q^2/r^2}{1-\mu} - \frac{\mu_* - Q_*^2/r^2}{1-\mu_*}\right\} \\ & - i\widehat{q}_0 A_U \lambda\{H + \partial_r(r\phi_*)\} + \frac{i\widehat{q}_0(-\nu)\lambda Q}{(1-\mu)r^2}(\psi + rH). \end{aligned} \tag{2.69}$$

The parallel-transport integrating factor for D_U has modulus one. This avoids estimating a derivative of ψ in the electromagnetic source.

The neutral terms in (2.69) are bounded by $C(N_\phi + N_m)r^{-3}$. Its remaining terms, including the $Q^2 - Q_*^2$ part of the geometric coefficient, are bounded using (2.66)– (2.67) by

$$C_{T,\delta_0}\left(\frac{\eta}{r^4} + \frac{\eta}{Lr}\right). \tag{2.70}$$

For example, the three absolute bounds are respectively

$$\frac{C|Q^2 - Q_*^2|}{r^4} \le C\frac{\eta}{r^4}, \qquad |\widehat{q}_0 A_U|\{|H| + C/r^2\} \le C\frac{\eta}{Lr},$$

and

$$\frac{C|\widehat{q}_0 Q|}{r^2}(|\psi| + r|H|) \le C\frac{\eta}{Lr}(1 + r^{-1}).$$

All constants here are independent of L, H, and q_0.

Along an incoming ray, $-\nu \ge c > 0$ by (2.61); hence its integral of (2.70) is at most

$$C\int_r^L \left(\frac{\eta}{s^4} + \frac{\eta}{Ls}\right) \mathrm{d}s \le C\frac{\eta}{r^3} + C\frac{\eta}{L}\log\frac{L}{r}.$$

The logarithm must be retained in a single derivative estimate. Its outgoing integral is nevertheless bounded by $C\eta$, since $L^{-1}\int_{L_0}^L \log(L/r)\,\mathrm{d}r \le 1$. The other outgoing integral is at most $C\eta/L_0^2$. Thus the double integral of the electric source is at most $C\eta$, uniformly in L. At the outer boundary

$$w(t,t) = O_T(\epsilon), \qquad \partial_V w(t,t) = \tfrac{1}{2}(\alpha/\lambda - H)_b(t).$$

Integrating (2.69) in U and then integrating $\partial_V w$ in V therefore gives

$$N_\phi \le C_T\epsilon + \frac{C}{L_0}(N_\phi + N_m) + C_{T,\delta_0}\eta, \tag{2.71}$$

$$\left|\frac{\theta}{\lambda} - \frac{\theta_*}{\lambda_*}\right| \le \frac{C_T\epsilon}{L} + \frac{CN_\phi}{r} + \frac{CN_m}{r^2} + \frac{C\eta}{r^3} + \frac{C\eta}{L}\log\frac{L}{r}. \tag{2.72}$$

Here $\theta/\lambda - \theta_*/\lambda_* = \lambda^{-1}\partial_V w - w/r$ was used.

The exact mass difference equation is

$$\begin{aligned} \partial_V \delta m = & -\frac{\lambda}{r}|\theta/\lambda|^2\delta m + \frac{\lambda(Q^2 - Q_*^2)}{2r^2} \\ & + \lambda(1-\mu_*)\,\mathrm{Re}\left[\overline{\theta_*/\lambda_*}(\theta/\lambda - \theta_*/\lambda_*)\right] \\ & + \frac{\lambda(1-\mu_*)}{2}|\theta/\lambda - \theta_*/\lambda_*|^2. \end{aligned} \tag{2.73}$$

Its backward integrating factor is bounded because $\int_{L_0}^{L} |\theta/\lambda|^2 \, \mathrm{d}r/r \le C/L_0$. The additional electric integral is at most

$$\int_{L_0}^{L} \frac{|Q^2 - Q_*^2|}{r^2} \, \mathrm{d}r \le C_{T,\delta_0}\eta/L_0. \tag{2.74}$$

Substitution of (2.72) gives

$$N_m \le \frac{C_T\epsilon + C_{T,\delta_0}\eta}{L_0} + \frac{C}{L_0}(N_\phi + N_\phi^2) + \frac{C}{L_0^2}N_m + \frac{C}{L_0^3}N_m^2. \tag{2.75}$$

One uses $L^{-1}\log(L/L_0) \le C/L_0$ for the boundary cross term. The electric logarithm is harmless because

$$\frac{1}{L}\log^2\frac{L}{L_0} \le \frac{C}{L_0}, \qquad \frac{1}{L^2}\int_{L_0}^{L} \log^2\frac{L}{r} \, \mathrm{d}r \le \frac{C}{L}.$$

By the segment inclusions above, these estimates hold on every $\mathcal{T}_p(\rho)$ with constants independent of ρ. Choose L_0 large enough to absorb the neutral terms. Under $N_\phi \le 1$, $N_m \le 4$ they yield

$$N_\phi(\rho) \le C(\epsilon + \eta), \qquad N_m(\rho) \le C(\epsilon + \eta)/L_0. \tag{2.76}$$

Once T is fixed, choose ϵ, η so that both $N_\phi(L)$ and the first right-hand side are below $1/2$. Continuity excludes a first cutoff with $N_\phi = 1$; hence (2.76) holds down to L_0.

The incoming trace remains an L^2 trace. Write $g_b = \phi_b - H + (\xi/\nu)_b$. At the outer boundary $\partial_U w(U,U) = -g_b(U)/2$. To transfer this derivative inward, use (2.69) in the form $w_{UV} = D_U w_V - i\widehat{q}_0 A_U w_V$ and integrate in V. The preceding derivative estimate gives $|w_V| \le C$ under the bootstrap. Thus the additional connection term has the same envelope (2.70). It follows that

$$|w_U(U,V) + g_b(U)/2| \le C(N_\phi + N_m)/r^2 + C\eta/r^3 + (C\eta/L)\log(L/r).$$

The incoming scalar difference is exactly

$$\frac{\xi}{\nu} - \frac{\theta_*}{\lambda_*} = \nu^{-1}w_U - w/r + i\widehat{q}_0 A_U \, r\phi/\nu.$$

The final correction is bounded by $C\eta/L$, because $r|\phi| \le r|H| + C$ on the outer annulus and $|A_U| \le C/r$. Hence

$$\left|\frac{\xi}{\nu} - \frac{\theta_*}{\lambda_*}\right|(U,V) \le C|g_b(U)| + \frac{CN_\phi}{r} + \frac{CN_m}{r^2} + C\eta/r^3 + (C\eta/L)\{1 + \log(L/r)\}. \tag{2.77}$$

On $r = L_0$, let t_0 be twice the rescaled cylinder clock. Then $\mathrm{d}t_0 = 2(-\nu)\,\mathrm{d}U \le \mathrm{d}U$. On any interval of inner-clock length at most a fixed K, the $L^2(\mathrm{d}t_0)$ norm of the difference in (2.77) is bounded by $C_{T,\delta_0,L_0,K}(\epsilon + \eta)$. The first term uses precisely the outer L^2 norm in (2.54); no pointwise bound for g_b is asserted.

Comparison on the fixed inner annulus. Renormalize both null labels to t_0 on $r = L_0$, retaining the same gauge. On $r_* \le r \le L_0$, the scalar bound gives

$$|\phi| \le |H| + C.$$

Maxwell's equation integrated from r outward to L_0, with the scalar L^2 mass bound, gives

$$|Q(r) - Q(L_0)| \le |\widehat{q}_0| \int_r^{L_0} s|\phi| \, |\theta/\lambda| \, \mathrm{d}s \le C|\widehat{q}_0|(1 + |H|) \le C\eta/L.$$

Thus (2.66) implies $|Q - Q_*| \le C\eta$ throughout this fixed annulus, and in particular $|Q| \le C$. Integrating the potential from its already controlled value at L_0 gives

$$|A_U| \le C, \qquad |\widehat{q}_0 A_U|(1 + |H|) \le C\eta/L \le C\eta.$$

The bounded change of null normalization only changes C. All inverse radii and inverse gaps of the actual and reference solutions are bounded on these comparison regions. The difference equations, with the reference incoming ratio equal to θ_*/λ_*, are exactly

$$\begin{aligned} D_U\left(\frac{\theta}{\lambda} - \frac{\theta_*}{\lambda_*}\right) = {} & \frac{-\nu}{r}\left[\frac{\xi}{\nu} - \frac{\theta_*}{\lambda_*} + \frac{\mu - Q^2/r^2}{1-\mu}\left(\frac{\theta}{\lambda} - \frac{\theta_*}{\lambda_*}\right)\right] \\ & + \frac{-\nu}{r}\left\{\frac{\mu - Q^2/r^2}{1-\mu} - \frac{\mu_* - Q_*^2/r^2}{1-\mu_*}\right\}\frac{\theta_*}{\lambda_*} \\ & + \frac{i\widehat{q}_0(-\nu)Q\phi}{(1-\mu)r} - i\widehat{q}_0 A_U \frac{\theta_*}{\lambda_*}, \end{aligned} \tag{2.78}$$

$$\begin{aligned} \partial_V\left(\frac{\xi}{\nu} - \frac{\theta_*}{\lambda_*}\right) = {} & -\frac{\lambda}{r}\left[\frac{\theta}{\lambda} - \frac{\theta_*}{\lambda_*} + \frac{\mu - Q^2/r^2}{1-\mu}\left(\frac{\xi}{\nu} - \frac{\theta_*}{\lambda_*}\right)\right] \\ & - \frac{\lambda}{r}\left\{\frac{\mu - Q^2/r^2}{1-\mu} - \frac{\mu_* - Q_*^2/r^2}{1-\mu_*}\right\}\frac{\theta_*}{\lambda_*} \\ & + \frac{i\widehat{q}_0\lambda Q\phi}{(1-\mu)r}. \end{aligned} \tag{2.79}$$

The difference of geometric coefficients is at most $C(|\delta m| + \eta)$, and every remaining charged source is at most $C\eta$. The covariant integrating factor in (2.78) is unitary. The mass equation (2.73) has an additional source at most $C\eta$ on this fixed annulus.

We specify the inner comparison region and its boundary norms. If $\rho_p = L_0$, the outer mass comparison already reaches the reference barrier and no inner comparison is needed. Otherwise the incoming and outgoing legs of $\mathcal{T}_p$ meet $r = L_0$ at two points. The portion of this timelike cylinder between them is the inner boundary interval. Normalize both null labels to t_0, twice its cylinder clock, and call the new labels U_0, V_0 in this paragraph. Then $U_0 = V_0 = t_0$ on $r = L_0$ and $-\nu_0 = \lambda_0 = 1/2$ there.

The inner sideways triangle is

$$\mathcal{S}_p = \{(U_0, V_0) : V_0(p) \le V_0 \le U_0 \le U_0(p)\}.$$

It lies in $\mathcal{T}_p$ by the same segment-inclusion argument and satisfies $\rho_p \le r \le L_0$. The optical estimate gives $c_* \le -\nu_0, \lambda_0 \le 1/2$ on its characteristic legs. In particular

$$U_0(p) - V_0(p) \le c_*^{-1}(L_0 - \rho_p) \le c_*^{-1} L_0.$$

Thus the inner boundary interval has clock length bounded by the fixed number $K = c_*^{-1} L_0$. The L^2 trace estimate obtained from (2.77) applies to this very interval; no extra inner window is postulated.

For $0 \le d \le L_0 - \rho_p$ define

$$\mathcal{S}_p(d) = \mathcal{S}_p \cap \{r \ge L_0 - d\}, \quad F(d) = \max_{\mathcal{S}_p(d)}\left|\frac{\theta}{\lambda} - \frac{\theta_*}{\lambda_*}\right|, \quad N(d) = \max_{\mathcal{S}_p(d)}|\delta m|.$$

These are continuous functions of d, by the radial truncation argument used above. Let e_θ, e_m denote the suprema of the two differences on the inner boundary interval, and let e_ξ denote the $L^2(\,\mathrm{d}t_0)$ norm of the incoming scalar difference on that interval. The outer estimates give $e_\theta + e_m + e_\xi \le C(\epsilon + \eta)$.

Under the temporary bound $F \le 1$, integration of (2.73) along the outward segment of each point, using the bounded reference coefficients, gives

$$N(d) \le Ce_m + C\eta + C\int_0^d F(s)\,\mathrm{d}s.$$

Integrating (2.79) along the same outward segments bounds the incoming difference by a bounded multiple of its boundary trace at the same U_0 label plus $C\eta + C\int_0^d (F+N)(s)\,\mathrm{d}s$. Insert this in the incoming integration of (2.78). The boundary-trace contribution obeys, with its actual incoming segment,

$$\begin{aligned}\int(-\nu_0)\,|\delta(\xi/\nu)_b(U_0)|\,\mathrm{d}U_0 &\le \left(\int(-\nu_0)\,\mathrm{d}U_0\right)^{1/2}\left(\int(-\nu_0)|\delta(\xi/\nu)_b(U_0)|^2\,\mathrm{d}U_0\right)^{1/2}\\ &\le \sqrt{d/2}\,e_\xi.\end{aligned}$$

Here the first integral is the traversed radial depth, and the second uses $-\nu_0 \le 1/2$ and the same boundary label $U_0 = t_0$. The inverse radii, gaps and expansions on the fixed annulus are bounded. All remaining kernels are therefore bounded, giving

$$F(d) + N(d) \le C(e_\theta + e_m + e_\xi + \eta) + C\int_0^d (F+N)(s)\,\mathrm{d}s.$$

Gronwall gives $F(d) + N(d) \le C(\epsilon + \eta)$ throughout each stopped interval. Choose the errors so that this bound and $F(0) = e_\theta$ are less than $1/2$. Continuity then excludes a first depth at which $F = 1$. The comparison therefore reaches p, the selected strict barrier point.

Parameter choice and conclusion. All comparison segments lie in the fixed $\mathcal{T}_p$ and its inner triangle. Choose L_0 for the neutral absorptions, then the compact reference barriers r_*, γ, their clock length $K = c_*^{-1}L_0$, and finally $T > 4/c_*$. The comparisons give $|m - m_*| \le C(\epsilon + \eta)$ with C now fixed. Choose η_* to satisfy the charge conditions and

$$C\eta_* < \min\{\gamma/4, r_*\gamma/8\}, \qquad C\epsilon < \min\{\gamma/4, r_*\gamma/8\},$$

then choose δ_* by (2.53) so that $L \ge L_0$. At p, either $m_* \le -\gamma$ or $1 - \mu_* \le \delta_0 - \gamma$. In the first case the mass closeness forces $m < 0$; in the second, using $|\mu - \mu_*| \le 2|m - m_*|/r_*$, it forces $1 - \mu < \delta_0$. Both contradict the actual geometry.

Failure of observation with constant C_* makes the two norms in (2.54) at most $C_T C_*^{-1/2}$ in their displayed units: the square of the original mass–scalar scaling denominator is comparable to $b^2(|\theta/\lambda|_b^2 + \mu_b^2)$, and adding charge only improves this normalized upper bound. Taking $C_* \ge C_T^2\epsilon^{-2}$ therefore proves (2.51). The fixed charged perturbation is absorbed against the strict barrier; it produces no additive energy error.

The AC justification uses the characteristic integral equations. On the compact region $r \ge \rho_p > 0$, the primary fields are continuous with nonvanishing denominators, and $D_U\phi = \xi/r$, $\partial_V\phi = \theta/r$ and the mixed radius derivative are continuous. Subtraction of the smooth radial reference gives (2.69) in integral form; Fubini and unitary transport justify its integrations. The positive continuous cylinder lapse gives C^1 null relabelings, and integrating the continuous tangential potential gives the boundary-parallel gauge. Changes of variables preserve the integral equations and boundary AC trace, as in Section 2.1. The only quadratic trace is the incoming L^2 trace transferred above with its bounded Jacobian. Uniform continuity gives the cutoff maxima and first-stop arguments. The axis is used only through the full outgoing constraint integrals and $m = Q = 0$ there.

□

The comparison retains both the complex phase in (2.65) and the scalar background H in every charged source. Regular-axis zero charge is essential to (2.58); the proposition does not cover an independently charged inner boundary.

2.5 Charged cylinder budgets and quantitative cooling

Integrating observation over overlapping windows gives duration-independent budgets and a quantitative decrease of Q_b. We include the endpoint intervals so the estimates also apply to unfinished stages.

Proposition 2.11 (Integrated charged observation and cooling)**.** *Fix the gap constant δ_0 and the corresponding constants $T_*, C_*, \delta_*, \eta_*$ of Proposition 2.10; decrease δ_* to be at most $1/2$ if necessary. Let $r = b > 0$ be a regular timelike cylinder on a boundary-clock interval $[0, T]$, with*

$$\mathrm{d}\tau = (-\nu_b)\,\mathrm{d}u = \lambda_b\,\mathrm{d}v_b.$$

Assume its normalized scalar datum is covariantly AC and

$$\mu_b + \left|\frac{\theta}{\lambda}\right|_b \le \delta_* \qquad \text{throughout } [0, T]. \tag{2.80}$$

For every full window $[t - T_* b, t + T_* b] \subset [0, T]$, assume that its inward characteristic comparison region belongs to the original development, that all the outgoing sections used there join the existing regular axis, and that $m \ge 0$ and $1 - \mu \ge \delta_0$ on those sections. At each such center also assume

$$|q_0| b\,(1 + |(\alpha/\lambda)_b(t)|) \le \eta_*. \tag{2.81}$$

Thus every full window satisfies exactly the hypotheses of Proposition 2.10. Put

$$Q_b(\tau) = \frac{b}{2}\left\{\left|\frac{\theta}{\lambda}\right|_b^2 - \log(1 - \mu_b)\right\}, \qquad \sigma = \mathrm{TV}^D_{[0,T]}((\alpha/\lambda)_b) = \int_0^T |D_\tau(\alpha/\lambda)_b|\,\mathrm{d}\tau.$$

There is a constant C, depending only on the observation constants, such that

$$\sup_{0\le\tau\le T} \frac{Q_b(\tau)}{b} + \int_0^T \left|\frac{\theta}{\lambda} - \frac{\xi}{\nu}\right|_b^2 \frac{\mathrm{d}\tau}{b} \le C\left\{\frac{Q_b(0)}{b} + \sigma^2\right\}, \tag{2.82}$$

$$\sup_{0\le\tau\le T}\left\{\frac{m_b}{b} + \left|\frac{\theta}{\lambda}\right|_b^2\right\} + \int_0^T \left\{\left|\frac{\theta}{\lambda}\right|_b^2 + \left|\frac{\xi}{\nu}\right|_b^2 + \mu_b^2 + \left(\frac{Q_b}{b}\right)^2\right\}\frac{\mathrm{d}\tau}{b} \le C\left\{\frac{Q_b(0)}{b} + \sigma^2\right\}. \tag{2.83}$$

The constant is independent of b, T/b, the size of the initial energy, and the value of the coupling subject to (2.81). The same estimates hold on every subinterval $[s, t] \subset [0, T]$, with $Q_b(s)$ and its own covariant variation on the right. In particular they hold on arbitrarily short or unfinished stages.

For each $d > 0$ there are $H_d, C_d > 0$, depending only on d and the observation constants, with the following property. If $\varepsilon > 0$, $Q_b(0)/b \le \varepsilon$, and the preceding hypotheses hold up to

$$T = H_d b/\varepsilon,$$

then

$$\frac{Q_b(T)}{b} \le d\varepsilon + C_d\sigma^2. \tag{2.84}$$

There is no required comparison between σ and ε, and no additive electric error term.

Proof. The exact charged identity (2.49), divided by b, is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{Q_b}{b}\right)+\frac{1}{2b}\left|\frac{\theta}{\lambda}-\frac{\xi}{\nu}\right|_b^2=\mathrm{Re}\left[\overline{(\theta/\lambda)_b}D_\tau(\alpha/\lambda)_b\right]. \tag{2.85}$$

The AC calculus above justifies this identity. Work in the parallel unitary frame, so boundary derivatives are ordinary derivatives.

The square-root bound and dissipation. Positivity of the Hawking mass gives $|\theta/\lambda|_b\le\sqrt{2Q_b/b}$. Apply (2.85) to $\sqrt{Q_b/b+\kappa}$, discard its nonnegative dissipation, and then let $\kappa\downarrow 0$. For $0\le s\le t\le T$ this gives

$$\sqrt{\frac{Q_b(t)}{b}}\le\sqrt{\frac{Q_b(s)}{b}}+\frac{1}{\sqrt{2}}\,\mathrm{TV}^D_{[s,t]}((\alpha/\lambda)_b). \tag{2.86}$$

Consequently

$$\sup_{[0,T]}\frac{Q_b}{b}\le 2\frac{Q_b(0)}{b}+\sigma^2.$$

Integrate (2.85) and drop its nonnegative final energy. Bounding the forcing by $\sqrt{2\sup Q_b/b}\,\sigma$ gives

$$\frac{1}{2}\int_0^T\left|\frac{\theta}{\lambda}-\frac{\xi}{\nu}\right|_b^2\frac{\mathrm{d}\tau}{b}\le\frac{Q_b(0)}{b}+\sqrt{2\sup_{[0,T]}Q_b/b}\,\sigma.$$

Young's inequality proves (2.82). This part uses only the exact identity and does not use the availability of an observation window.

Counting full windows and endpoint intervals. Let ω be the finite positive measure

$$\mathrm{d}\omega=|D_\tau(\alpha/\lambda)_b|\,\mathrm{d}\tau,\qquad \omega([0,T])=\sigma.$$

If $T\ge 2T_*b$, integrate (2.51) over the centers $t\in[T_*b,T-T_*b]$. A given time belongs to windows whose centers occupy a set of length at most $2T_*b$. The squared variation term has the precise bound

$$\int_{T_*b}^{T-T_*b}\omega([t-T_*b,t+T_*b])^2\,\mathrm{d}t\le 2T_*b\,\sigma^2. \tag{2.87}$$

Indeed, write the square as an integral against $\omega\otimes\omega$ and use Tonelli: the centers whose windows contain both prescribed points form an interval of length at most $2T_*b$. Integrating observation, dividing by b^2, and using these two multiplicity bounds yields

$$\begin{aligned}\int_{T_*b}^{T-T_*b}\left\{\left|\frac{\theta}{\lambda}\right|_b^2+\mu_b^2\right\}\frac{\mathrm{d}t}{b}\\ \le 2C_*T_*\left\{\int_0^T\left|\frac{\theta}{\lambda}-\frac{\xi}{\nu}\right|_b^2\frac{\mathrm{d}\tau}{b}+\sigma^2\right\}.\end{aligned} \tag{2.88}$$

The uncaptured endpoint intervals have total length at most $2T_*b$. Since $\mu_b\le\delta_*$ and $-\log(1-\mu_b)\ge\mu_b$, one has

$$\left|\frac{\theta}{\lambda}\right|_b^2\le 2Q_b/b,\qquad \mu_b^2\le\delta_*\mu_b\le 2\delta_*Q_b/b.$$

Their contribution is therefore at most $C \sup_{[0,T]} Q_b/b$, which is controlled by (2.82). If $T < 2T_* b$, the whole interval has this length bound and the same estimate applies without using any full window. In every case,

$$\int_0^T \left\{ \left| \frac{\theta}{\lambda} \right|_b^2 + \mu_b^2 \right\} \frac{\mathrm{d}\tau}{b} \le C \left\{ \frac{Q_b(0)}{b} + \sigma^2 \right\}. \tag{2.89}$$

For the incoming ratio use the complex triangle inequality,

$$\left| \frac{\xi}{\nu} \right|_b^2 \le 2 \left| \frac{\theta}{\lambda} \right|_b^2 + 2 \left| \frac{\theta}{\lambda} - \frac{\xi}{\nu} \right|_b^2.$$

On the fixed weak regime,

$$\frac{Q_b}{b} \asymp \left| \frac{\theta}{\lambda} \right|_b^2 + \mu_b, \qquad \left(\frac{Q_b}{b} \right)^2 \le C \left\{ \left| \frac{\theta}{\lambda} \right|_b^2 + \mu_b^2 \right\}.$$

The last inequality uses $|\theta/\lambda|_b \le \delta_*$; its fourth power is bounded by a fixed multiple of its square. Also $m_b/b = \mu_b/2 \le Q_b/b$. These observations prove (2.83).

Repeating the proof on a subinterval uses its inherited full-window hypotheses, its own endpoint intervals, and its own variation. Thus the budgets hold before a stage is completed.

Cooling at the time scale b/ε. Let C be the constant controlling the integral of $(Q_b/b)^2$ in (2.83). For $T = H_d b/\varepsilon$, averaging that integral over $[T/2, T]$ provides a time s in this interval such that

$$\begin{aligned} \frac{Q_b(s)}{b} &\le \left(\frac{2C\varepsilon}{H_d} (\varepsilon + \sigma^2) \right)^{1/2} \\ &\le \left(\frac{2C}{H_d} \right)^{1/2} (\varepsilon + \sqrt{\varepsilon}\, \sigma). \end{aligned}$$

The square-root inequality (2.86) then gives

$$\frac{Q_b(T)}{b} \le 2 \frac{Q_b(s)}{b} + \sigma^2.$$

For an explicit choice, take

$$H_d = \frac{18C}{d^2}, \qquad C_d = 1 + \frac{d}{3}. \tag{2.90}$$

Then $(2C/H_d)^{1/2} = d/3$. Using $2\sqrt{\varepsilon}\, \sigma \le \varepsilon + \sigma^2$ in the two preceding inequalities gives, term by term,

$$\begin{aligned} \frac{Q_b(T)}{b} &\le \frac{2d}{3} \varepsilon + \frac{2d}{3} \sqrt{\varepsilon}\, \sigma + \sigma^2 \\ &\le d\varepsilon + \left(1 + \frac{d}{3} \right) \sigma^2. \end{aligned}$$

This proves (2.84) without assuming that the variation error is smaller than the initial energy.

The time scale b/ε reflects the quadratic mass term in observation versus the linear mass term in Q_b/b. Here ε is an upper envelope for the initial energy. The calculation imposes no smallness on $\sigma/\sqrt{\varepsilon}$ and adds no electric error. □

Corollary 2.12 (Fixed fractional decrease of the actual energy)**.** *Fix $0 < d < 1$ and suppose the hypotheses of Proposition 2.11 hold on the stated interval. If $Q_b(0) > 0$, let*

$$T = H_d \frac{b^2}{Q_b(0)}, \qquad \sigma = \mathrm{TV}^D_{[0,T]}((\alpha/\lambda)_b). \tag{2.91}$$

Then

$$Q_b(T) \le d Q_b(0) + \left(1 + \frac{d}{3}\right) b\sigma^2. \tag{2.92}$$

If $Q_b(0) = 0$, every available interval instead satisfies

$$Q_b(T) \le \frac{b}{2}\left[\mathrm{TV}^D_{[0,T]}((\alpha/\lambda)_b)\right]^2. \tag{2.93}$$

All constants are independent of b, $Q_b(0)/b$, and the initial scalar amplitude, subject to the physical smallness condition (2.81).

Proof. For $Q_b(0) > 0$, take $\varepsilon = Q_b(0)/b$ in (2.84) and multiply by b. For zero initial energy, square (2.86) with $s = 0$. □

The error in this corollary can also be written directly in terms of the outgoing derivative α' defined at the start of the manuscript. The characteristic equations give, on every proper cylinder,

$$D_\tau(\alpha/\lambda)_b = \alpha_b' + \frac{\mu_b - Q_{\mathrm{cyl}}^2/b^2}{(1-\mu_b)b}(\theta/\lambda)_b + \frac{i q_0 Q_{\mathrm{cyl}} \phi_b}{(1-\mu_b)b}.$$

Consequently

$$\sigma \le \int_0^T \left\{ |\alpha_b'| + \frac{|\mu_b - Q_{\mathrm{cyl}}^2/b^2|}{(1-\mu_b)b}|\theta/\lambda|_b + \frac{|q_0 Q_{\mathrm{cyl}}\phi_b|}{(1-\mu_b)b} \right\} \mathrm{d}\tau. \tag{2.94}$$

Lemmas 2.1 and 2.2 justify this trace and chain rule. Electric terms cancel in the energy balance but remain in the variation error, estimated in Section 2.6 through the equivalent renormalized identity (2.126).

2.5.1 Application under the terminal-cone hypothesis

The dependencies are as follows. Terminal equivalence (Proposition 2.3) gives finiteness of the positive terminal integral and terminal mass decay. The scalar and mass bounds give $r|\phi| \to 0$ and the independent core gap (Lemma 2.4 and Proposition 2.8). These precede observation; the latter derives its optical bounds only down to the positive reference barrier. No uniform interior blueshift is a premise.

The adaptive exterior gives small boundary data on the first cylinder. On a later partial cylinder, the identity alone gives

$$\sup \frac{Q_b}{b} \le 2\varepsilon_n + \sigma_n^2, \qquad \mu_b \le 2Q_b/b, \qquad |\theta/\lambda|_b \le \sqrt{2Q_b/b}.$$

Small stopped budgets therefore preserve the observation hypotheses without using a future completed window. Exterior reconstruction bounds $|\phi| \le M_0$, so $|q_0|b(1 + |\alpha/\lambda|_b) \le |q_0|b(1 + M_0 + F)$ is small on a sufficiently late starting row. Section 2.7 proves the strict improvements and availability of the windows.

The construction uses an envelope $Q_{b_n}(0)/b_n \le \varepsilon_n$ and clock $H_d b_n/\varepsilon_n$, obtaining $Q_{b_n}(T)/b_n \le d\varepsilon_n + C_d\sigma_n^2$. This is a fraction of the envelope, whereas Corollary 2.12 gives a fraction of the actual energy at clock $H_d b^2/Q_b(0)$. The envelope recurrence controls the sum of prescribed clock lengths, and the partial-stage budgets control the variation before each stage ends. Uniform interior blueshift follows after continuation.

2.6 The charged renormalized derivative and exterior interactions

We estimate the derivative source and optical integrals uniformly over long cooling intervals. We retain the conventions $\mu = 2m/r$, $D = \mathrm{d} + iq_0 A$, $A_v = 0$; Q is electric charge, Q_b is boundary energy, and J will denote the integral of an incoming derivative envelope.

The estimates here are related to the characteristic variation method of Christodoulou [6] and the charged area estimates of An–Tan [20, Section 5]. We express the curvature source as a characteristic integral of a renormalized derivative and substitute this expression into the differentiated transport equation. The order of integration retains the radial factors needed for integrability at the axis. Replacing this integral by a pointwise bound would lose that information. The cubic correction in (2.95) and the full complex quantity (2.96) are required by the exact equations derived below.

2.6.1 The complex curvature combination is necessary

Define

$$j = (1-\mu)\lambda\left[\alpha' - \frac{1}{r}\left|\frac{\theta}{\lambda}\right|^2\frac{\theta}{\lambda}\right], \tag{2.95}$$

$$\mathcal{C} = \mu + (1-\mu)\frac{\theta}{\lambda}\frac{\overline{\xi}}{\nu}. \tag{2.96}$$

The quantity $\mathcal{C}$ is generally complex, although gauge invariant. Keeping only its real part is insufficient even when $q_0 = Q = 0$. The additional phase term in the derivative equation is $-i(\theta/\lambda)\operatorname{Im}\mathcal{C}$; it does not vanish for two independent real scalar components.

Lemma 2.13 (Exact charged renormalized identities)**.** *In the normalization*

$$\begin{aligned}
r_{uv} &= \frac{\nu\lambda}{r}\frac{\mu - Q^2/r^2}{1-\mu},\\
m_v &= \frac{(1-\mu)|\theta|^2}{2\lambda} + \frac{Q^2\lambda}{2r^2}, & m_u &= \frac{(1-\mu)|\xi|^2}{2\nu} + \frac{Q^2\nu}{2r^2},\\
Q_v &= -q_0 r\operatorname{Im}(\bar\phi\theta), & Q_u &= q_0 r\operatorname{Im}(\bar\phi\xi),
\end{aligned}$$

the exact system is

$$D_u j = -\frac{|\xi|^2}{\nu r}j + \frac{\nu\theta}{r^2}(\mathcal{C} + 2\overline{\mathcal{C}}) + \mathcal{E}_j, \tag{2.97}$$

$$\partial_v(r\mathcal{C}) = r\frac{\overline{\xi}}{\nu}j + \frac{\lambda Q^2}{r^2} - iq_0 Q\theta\bar\phi, \tag{2.98}$$

$$D_u\left(\frac{\lambda j}{1-\mu}\right) = \frac{\nu\lambda\theta}{(1-\mu)r^2}(\mathcal{C} + 2\overline{\mathcal{C}}) + \frac{\lambda}{1-\mu}\mathcal{E}_j, \tag{2.99}$$

where

$$\mathcal{E}_j = \frac{(-\nu)\lambda}{r^2}\left[5\frac{Q^2}{r^2}\frac{\theta}{\lambda} + iq_0 Q\left(3\frac{\theta}{\lambda} + \phi\right)\right] + \frac{iq_0(-\nu)}{r}Q_v\phi. \tag{2.100}$$

No derivative of α' occurs in this source.

Proof. The radial identities are

$$\begin{aligned} D_r(\theta/\lambda) &= \frac{j}{(1-\mu)\lambda} - \frac{1-|\theta/\lambda|^2}{r}\frac{\theta}{\lambda}, \\ D_r(\xi/\nu) &= -\frac{1}{r}\frac{\theta}{\lambda} - \frac{\mu - Q^2/r^2}{(1-\mu)r}\frac{\xi}{\nu} + \frac{iq_0Q\phi}{(1-\mu)r}, \\ \partial_r\mu &= \frac{(1-\mu)|\theta/\lambda|^2 + Q^2/r^2 - \mu}{r}, \\ \partial_r \log(-\nu) &= \frac{\mu - Q^2/r^2}{(1-\mu)r}. \end{aligned}$$

Here $D_r = \lambda^{-1}D_v$. Differentiating $r\mathcal{C}$, all scalar terms other than the derivative source cancel. The result is

$$\partial_r(r\mathcal{C}) = r\frac{\overline{\xi}}{\nu}\frac{j}{\lambda} + \frac{Q^2}{r^2} - iq_0Q\frac{\theta}{\lambda}\bar{\phi},$$

which proves (2.98).

For the incoming calculation use

$$\begin{aligned} D_u(\alpha/\lambda) &= \frac{-\nu}{r}\left[\frac{\mu - Q^2/r^2}{1-\mu}\frac{\theta}{\lambda} + \frac{iq_0Q\phi}{1-\mu}\right], \\ D_u(\theta/\lambda) &= \frac{-\nu}{r}\left[\frac{\xi}{\nu} + \frac{\mu - Q^2/r^2}{1-\mu}\frac{\theta}{\lambda} + \frac{iq_0Q\phi}{1-\mu}\right], \\ [D_u, D_v]\phi &= \frac{2iq_0(-\nu)\lambda Q}{(1-\mu)r^2}\phi. \end{aligned}$$

The cubic derivative must retain both complex contributions:

$$D_u\left(|\theta/\lambda|^2\theta/\lambda\right) = 2\,\mathrm{Re}\left(\overline{\theta/\lambda}D_u(\theta/\lambda)\right)\frac{\theta}{\lambda} + |\theta/\lambda|^2 D_u(\theta/\lambda).$$

The neutral mixed terms become

$$2\,\mathrm{Re}\left(\overline{\theta/\lambda}\frac{\xi}{\nu}\right)\frac{\theta}{\lambda} + |\theta/\lambda|^2\frac{\xi}{\nu} = \left(\frac{\theta}{\lambda}\right)^2\frac{\overline{\xi}}{\nu} + 2|\theta/\lambda|^2\frac{\xi}{\nu}.$$

Together with the mass term this is precisely $(\theta/\lambda)(\mathcal{C} + 2\bar{\mathcal{C}})/(1-\mu)$. In the electromagnetic terms, differentiation of Q^2/r^2 initially produces a multiple of $QQ_r(\theta/\lambda)$. It cancels against the real part in the cubic derivative, because

$$Q_r = -q_0 r\,\mathrm{Im}\left(\bar{\phi}\frac{\theta}{\lambda}\right), \qquad \mathrm{Re}\left(\overline{\theta/\lambda}\frac{iq_0Q\phi}{1-\mu}\right) = -\frac{QQ_r}{(1-\mu)r}.$$

The remaining terms give (2.100) and (2.99). Finally, $\partial_u \log(\lambda/(1-\mu)) = |\xi|^2/(\nu r)$ gives (2.97). The common representative and AC product rules of Lemma 2.2 justify these characteristic identities on proper positive-radius domains: their sources have majorants $C(1+|g(v)|)$, with $g \in L^1$. □

2.6.2 Scalar and charge profiles from the exterior derivative envelope

Let D have outgoing rows $v_b(u) \le v \le 0$, with $r = b(u)$ at the inner boundary. The earlier geometric argument supplies a global gap $1-\mu \ge \delta$, where $\delta > 0$ denotes its actual lower bound, including the

factor $1/2$ in Proposition 2.8. On a sufficiently late triangle, $Q^2/r^2 \le \mu/2$. In the stopped exterior assume

$$c \le \lambda \le \Lambda, \qquad |\theta/\lambda| \le F < 1, \qquad \frac{\mu}{1-\mu} \le \varepsilon, \qquad 1 - F^2 > \max\{\varepsilon, 1/2\}. \tag{2.101}$$

Thus

$$0 \le \frac{\mu - Q^2/r^2}{1-\mu} \le \varepsilon, \qquad \frac{-\nu(u,r)}{-\nu(u,x)} \le (r/x)^{\varepsilon} \quad (r \ge x).$$

The global relative-charge bound can itself be checked from

$$\frac{Q(u,r)^2}{r^2} \le \frac{q_0^2}{\delta} \left(\sup_{0<x\le r} x|\phi(u,x)| \right)^2 \mu(u,r).$$

Indeed, apply Cauchy–Schwarz to the outgoing Maxwell equation and use $\int_0^r (1-\mu)|\theta/\lambda|^2 \,\mathrm{d}x \le 2m(u,r)$. The previously established vanishing of $r|\phi|$ therefore ensures the required nonnegativity of the radial expansion coefficient.

Let $R = r(u,0)$, let R_0 be its entry value, and stop also at $R \ge R_0/2$. Suppose $g \in L^1$ majorizes $|\lambda j/(1-\mu)|$, and set $J = \int g(v)\,\mathrm{d}v$. The covariant outgoing variation of α/λ on one row satisfies

$$\mathrm{TV}_{[b,R]}(\alpha/\lambda) \le c^{-1} J + \int_b^R |\theta/\lambda|^3 \frac{\mathrm{d}r}{r}.$$

In the gauge $A_v = 0$ the exact radial identity is

$$r\frac{\theta}{\lambda}(r) = b\frac{\theta_b}{\lambda_b} + \int_b^r x\partial_x(\alpha/\lambda)\,\mathrm{d}x.$$

It gives

$$\begin{aligned} \int_b^R |\theta/\lambda| \frac{\mathrm{d}r}{r} &\le |\theta_b/\lambda_b| + \mathrm{TV}_{[b,R]}(\alpha/\lambda), \\ (1-F^2)\,\mathrm{TV}_{[b,R]}(\alpha/\lambda) &\le c^{-1}J + F^2|\theta_b/\lambda_b|. \end{aligned}$$

Consequently, if $J \le J_{\mathrm{stop}}$, then

$$|\phi(u,r)| \le M_0 := 1 + \sup_{R_0/2 \le R \le R_0} |\phi|_{v=0} + \frac{F + c^{-1}J_{\mathrm{stop}}}{1-F^2}. \tag{2.102}$$

This bounds the stopped exterior without a scalar bound in the excluded core.

The core supplies a boundary charge estimate through its mass. For $0 < x < b$, the global gap and the outgoing mass equation give

$$|\phi(u,x)| \le |\phi_b| + \left(\int_x^b (1-\mu)|\theta/\lambda|^2 \,\mathrm{d}s \right)^{1/2} \left(\int_x^b \frac{\mathrm{d}s}{(1-\mu)s^2} \right)^{1/2} \le M_0 + \sqrt{\frac{2m_b}{\delta x}}.$$

Maxwell's equation and Cauchy–Schwarz now yield

$$\begin{aligned} |Q(u,v_b)|^2 &\le q_0^2 \left(\int_0^b (1-\mu)|\theta/\lambda|^2 \,\mathrm{d}x \right) \left(\int_0^b \frac{x^2|\phi|^2}{1-\mu} \,\mathrm{d}x \right) \\ &\le C_\delta q_0^2 m_b (M_0^2 b^3 + m_b b^2). \end{aligned} \tag{2.103}$$

In particular,

$$\frac{|Q(u,v_b)|}{b^2} \le C_\delta |q_0| \left[M_0 \sqrt{m_b/b} + m_b/b \right]. \tag{2.104}$$

Suppose the stage budgets satisfy $m_b/b \le e_n \le F^2$. Integrating Maxwell's equation from b to r in the exterior and using (2.102) gives

$$\frac{|Q(u,r)|}{r^2} \le C_\delta |q_0| M_0 F \left(\frac{b}{r}\right)^2 + \frac{|q_0| M_0 F}{r^2} \int_b^r x \, \mathrm{d}x \le C_\delta |q_0| M_0 F. \tag{2.105}$$

The stopped construction below obtains $e_n \le F^2$ directly from its recurrence and variation stop, before using the interaction estimate.

Set

$$\zeta^2 = q_0^2 M_0^2 R_0^2. \tag{2.106}$$

The terminal scalar estimate gives $M_0^2 \le C[1 + \log(1/R_0)]$ for late entries, with fixed stopping parameters. Hence $\zeta \to 0$ as $R_0 \to 0$.

2.6.3 The charged additions to the radial reconstruction

The radial equation for θ/λ has the real damping coefficient $1 - |\theta/\lambda|^2$. Split it into its boundary and derivative-source parts, denoted by superscripts b and j. Split ξ/ν accordingly, with one additional electromagnetic part. Integrating the radial equations gives

$$\begin{aligned}
|(\theta/\lambda)^b(r)| &\le |\theta_b/\lambda_b| (b/r)^{1-F^2}, \\
|(\theta/\lambda)^j(r)| &\le \int_b^r (x/r)^{1-F^2} \frac{1}{\lambda(x)^2} \left| \frac{\lambda j}{1-\mu}(x) \right| \mathrm{d}x, \qquad (2.107) \\
|(\xi/\nu)^b(r)| &\le \frac{-\nu_b}{-\nu(r)} \left(|\xi_b/\nu_b| + \frac{|\theta_b/\lambda_b|}{1 - F^2 - \varepsilon} \right), \qquad (2.108) \\
|(\xi/\nu)^j(r)| &\le \frac{1}{1-F^2} \int_b^r \frac{r-x}{r} \frac{1}{\lambda(x)^2} \left| \frac{\lambda j}{1-\mu}(x) \right| \mathrm{d}x. \qquad (2.109)
\end{aligned}$$

For (2.108), use $-\nu$ as the integrating factor and $(-\nu(x))/(-\nu_b) \le (x/b)^\varepsilon$. For (2.109), discard only the integrating factor whose modulus is at most one and use Tonelli:

$$\int_x^r (x/s)^{1-F^2} \frac{\mathrm{d}s}{s} = \frac{1 - (x/r)^{1-F^2}}{(1-F^2)} \le \frac{r-x}{(1-F^2)r}.$$

The last inequality uses $0 < 1 - F^2 \le 1$. The charged part satisfies

$$|(\xi/\nu)^{\mathrm{em}}(r)| \le \frac{|q_0|}{\delta} \int_b^r \frac{|Q\phi|(x)}{x} \mathrm{d}x \le C q_0^2 M_0^2 F r^2. \tag{2.110}$$

The integrating factor $(-\nu(x))/(-\nu(r))$ is at most one, which justifies the first inequality. Unitary parallel transport does not change it.

The full radial reconstruction is

$$\begin{aligned}
\mathcal{C}(u,r) = \frac{b}{r}\mathcal{C}_b &+ \frac{1}{r} \int_b^r \frac{(1-\mu)x}{\lambda^2} \frac{\overline{\xi}}{\nu} \frac{\lambda j}{1-\mu} \mathrm{d}x \\
&+ \frac{1}{r} \int_b^r \left[\frac{Q^2}{x^2} - i q_0 Q \frac{\theta}{\lambda} \bar{\phi} \right] \mathrm{d}x.
\end{aligned} \tag{2.111}$$

By (2.105), the last term of (2.111) is bounded by $C q_0^2 M_0^2 F^2 r^2$. The contribution containing $(\xi/\nu)^{\mathrm{em}} j$ is estimated separately.

Lemma 2.14 (All norms on an incoming connector)**.** *Fix $0 < \rho < 1$. Let an incoming connector have constant label v, initial radius B, and final radius ρB. Parametrize it by $\tau = B - b$, where $b = r$ on the connector. Suppose that on its complete inward outgoing sections $1 - \mu \geq \delta > 0$, that $0 \leq Q^2/b^2 \leq \mu$, and that $|\phi_b| \leq M_0$, with $M_0 \geq 1$. If*

$$\frac{m(0)}{B} + \left|\frac{\theta}{\lambda}(0)\right|^2 \leq e_n, \qquad 0 \leq e_n \leq 1, \qquad q_0^2 M_0^2 B^2 \leq 1,$$

then, with a constant depending only on ρ, δ,

$$\sup_{\text{connector}} \left(\frac{m_b}{b} + \left|\frac{\theta_b}{\lambda_b}\right|^2\right) \leq C_{\rho,\delta} e_n, \tag{2.112}$$

$$\int_{\text{connector}} \left(\left|\frac{\theta_b}{\lambda_b}\right|^2 + \left|\frac{\xi_b}{\nu_b}\right|^2 + \mu_b^2\right) \frac{\mathrm{d}\tau}{b} \leq C_{\rho,\delta} e_n. \tag{2.113}$$

The normalized observation energy on the final sphere is at most $C_{\rho,\delta} e_n$. Moreover,

$$\int_{\text{connector}} |D_\tau(\alpha/\lambda)_b| \, \mathrm{d}\tau \leq C_{\rho,\delta} e_n^{3/2} + C_{\rho,\delta} q_0^2 M_0^2 B^2 \sqrt{e_n}. \tag{2.114}$$

The same conclusions hold for every partial connector.

Proof. Since $\mathrm{d}\tau = (-\nu_b)\,\mathrm{d}u = -\,\mathrm{d}b$, the exact mass equation is

$$\frac{\mathrm{d}m_b}{\mathrm{d}\tau} = -\frac{1-\mu_b}{2}\left|\frac{\xi_b}{\nu_b}\right|^2 - \frac{Q(u,v_b)^2}{2b^2}. \tag{2.115}$$

Mass is nonnegative on a complete outgoing section. Integration and $b \geq \rho B$ give

$$\frac{m_b}{b} \leq \frac{e_n}{\rho}, \quad \int_{\text{connector}} \left|\frac{\xi_b}{\nu_b}\right|^2 \frac{\mathrm{d}\tau}{b} \leq \frac{2e_n}{\delta\rho}, \quad \int_{\text{connector}} \frac{Q(u,v_b)^2}{b^2}\,\mathrm{d}\tau \leq 2Be_n. \tag{2.116}$$

The core charge estimate (2.104) gives

$$\frac{|Q(u,v_b)|}{b^2} \leq C_\delta |q_0| \left(M_0\sqrt{m_b/b} + m_b/b\right) \leq C_{\rho,\delta}|q_0| M_0 \sqrt{e_n}.$$

The exact scalar transport along the connector is

$$D_\tau \frac{\theta_b}{\lambda_b} = \frac{1}{b}\left[\frac{\xi_b}{\nu_b} + \frac{\mu_b - Q(u,v_b)^2/b^2}{1-\mu_b}\frac{\theta_b}{\lambda_b} + \frac{iq_0 Q(u,v_b)\phi_b}{1-\mu_b}\right]. \tag{2.117}$$

After unitary parallel transport, the real coefficient has integral at most $\delta^{-1}\log(1/\rho)$. Its integrating factor and inverse are bounded, while Cauchy–Schwarz and the charge bound give

$$\int_{\text{connector}} \left|\frac{\xi_b}{\nu_b}\right| \frac{\mathrm{d}\tau}{b} \leq \left(\frac{2e_n}{\delta\rho}\right)^{1/2} \left(\log\frac{1}{\rho}\right)^{1/2},$$

$$\int_{\text{connector}} \frac{|q_0 Q(u,v_b)\phi_b|}{(1-\mu_b)b}\,\mathrm{d}\tau \leq C_{\rho,\delta} q_0^2 M_0^2 \sqrt{e_n} \int_{\rho B}^{B} b\,\mathrm{d}b \leq C_{\rho,\delta} q_0^2 M_0^2 B^2 \sqrt{e_n}.$$

Variation of constants gives $\sup|\theta_b/\lambda_b| \leq C_{\rho,\delta}\sqrt{e_n}$. Together with $\int \mathrm{d}\tau/b \leq \log(1/\rho)$ and $\mu_b \leq 2e_n/\rho$, this proves (2.112)–(2.113). The energy bound follows from $-\log(1-\mu_b) \leq \mu_b/\delta$.

For the normalized datum the ξ/ν terms cancel between ϕ and θ/λ, giving

$$D_\tau(\alpha/\lambda)_b = \frac{1}{b}\left[\frac{\mu_b - Q(u,v_b)^2/b^2}{1-\mu_b}\frac{\theta_b}{\lambda_b} + \frac{iq_0 Q(u,v_b)\phi_b}{1-\mu_b}\right].$$

The two terms have the bounds in (2.114). All calculations apply unchanged to a partial connector. □

Lemma 2.15 (Source kernels with moving radius)**.** *The preceding connector estimates control every boundary term in Proposition 2.16, with constants independent of the radius, position, and number of preceding stages.*

Proof. Fix a surviving exterior incoming label s and write $r_s = r(u,s)$, $\nu_s = \nu(u,s)$. Radial optical comparison gives

$$r_s \geq b, \qquad 1 \leq \frac{-\nu_s}{-\nu_b} \leq (r_s/b)^\varepsilon.$$

Use $\varepsilon < 1$, which follows from (2.101), and $\mathrm{d}\tau = (-\nu_b)\,\mathrm{d}u = -\,\mathrm{d}b$:

$$\int \frac{b^3}{r_s^4}\left(\frac{\nu_s}{\nu_b}\right)^2 \mathrm{d}\tau \leq \int \frac{1}{b}\left(\frac{b}{r_s}\right)^{4-2\varepsilon} \mathrm{d}\tau \leq \log(1/\rho), \tag{2.118}$$

$$\int \frac{b}{r_s^2}\,\mathrm{d}\tau \leq \log(1/\rho), \tag{2.119}$$

$$\int \frac{b(-\nu_s)}{r_s^2}\,\mathrm{d}u \leq \int \frac{1}{b}\left(\frac{b}{r_s}\right)^{2-\varepsilon} \mathrm{d}\tau \leq \log(1/\rho). \tag{2.120}$$

All integrals are over the surviving part of the label, bounded above by the full connector. The first boundary source satisfies

$$\int \left|\frac{\theta_b}{\lambda_b}\right|\left(\mu_b + \left|\frac{\theta_b}{\lambda_b}\right|\left|\frac{\xi_b}{\nu_b}\right|\right)\frac{\mathrm{d}\tau}{b} \leq C_{\rho,\delta} e_n^{3/2} \leq C_{\rho,\delta} e_n.$$

For the second source, the mass part satisfies $\int b\mu_b(-\nu_s)r_s^{-2}\,\mathrm{d}u \leq C_{\rho,\delta}e_n$. Its mixed part satisfies

$$\begin{aligned}\int b\left|\frac{\theta_b}{\lambda_b}\right|\left|\frac{\xi_b}{\nu_b}\right|\frac{-\nu_s}{r_s^2}\,\mathrm{d}u &\leq \sup\left|\frac{\theta_b}{\lambda_b}\right|\left(\int\left|\frac{\xi_b}{\nu_b}\right|^2\frac{\mathrm{d}\tau}{b}\right)^{1/2}\left(\int\frac{b^3}{r_s^4}\left(\frac{\nu_s}{\nu_b}\right)^2\mathrm{d}\tau\right)^{1/2}\\ &\leq C_{\rho,\delta}e_n.\end{aligned}$$

After integration against $g(s)$ this costs $C_{\rho,\delta}e_n J$. For the third source, Cauchy–Schwarz gives

$$\int \frac{|\xi_b/\nu_b| + C|\theta_b/\lambda_b|}{r_s}\,\mathrm{d}\tau \leq C_{\rho,\delta}\sqrt{e_n}.$$

Its contribution is $C_{\rho,\delta}FJ\sqrt{e_n}$. Equation (2.120) bounds the optical boundary mass term by $C_{\rho,\delta}e_n$. The interior terms use only radial ordering and positive area measure. Finally,

$$\mathrm{d}\tau = (-\nu_b)\,\mathrm{d}u \leq (-\nu(u,0))\,\mathrm{d}u = -\,\mathrm{d}R, \qquad \int b\,\mathrm{d}\tau \leq \int R(-\,\mathrm{d}R) \leq R_0^2/2.$$

These are the connector source, optical, and electromagnetic bounds used below. □

Proposition 2.16 (Charged source and optical estimates)**.** *Assume the stopped exterior hypotheses above. Suppose the cylindrical budgets e_n control*

$$\sup\left(\frac{m_b}{b}+|\theta_b/\lambda_b|^2\right), \qquad \int\left[|\theta_b/\lambda_b|^2+|\xi_b/\nu_b|^2+\mu_b^2\right]\frac{\mathrm{d}\tau}{b}.$$

Include each connector with the budget of its preceding cylinder. For

$$N=\iint_D\left|D_u\left(\frac{\lambda j}{1-\mu}\right)\right|\mathrm{d}u\,\mathrm{d}v,$$

one has

$$N\le C\sum_n e_n+CJ\sum_n e_n+CFJ\sum_n\sqrt{e_n}+CFJ^2+C\zeta^2(1+J). \tag{2.121}$$

Moreover,

$$\sup_v\int_{D\cap\{v=\text{constant}\}}\frac{(-\nu)\mu}{(1-\mu)r}\,\mathrm{d}u\le C\sum_n e_n+CJ^2+C\zeta^2, \tag{2.122}$$

$$\log\frac{-\nu(u,0)}{-\nu_b}\le C\left(\frac{m_b}{b}+|\theta_b/\lambda_b|^2+J^2+\zeta^2\right). \tag{2.123}$$

The constants depend only on the fixed geometric and stopping parameters, including Λ and the connector ratio. They are independent of q_0, M_0, R_0, the number of stages, and their duration.

Proof. For the scalar interaction terms use $|\mathcal{C}+2\bar{\mathcal{C}}|\le 3|\mathcal{C}|$. We first display the two-label estimate. Put $r_s=r(u,s)$. Since radial source measure converts according to

$$\frac{1}{\lambda^2}\left|\frac{\lambda j}{1-\mu}\right|\mathrm{d}r=\frac{1}{\lambda}\left|\frac{\lambda j}{1-\mu}\right|\mathrm{d}v\le c^{-1}g(v)\,\mathrm{d}v,$$

(2.109) and $r_s-r_t\le\Lambda(s-t)$ give

$$r_s|(\xi/\nu)^j(u,s)|\le\frac{\Lambda}{(1-F^2)c}\int_{v_b(u)}^s(s-t)g(t)\,\mathrm{d}t.$$

After substitution into (2.111), the pair part of the source in (2.99) is at most

$$\frac{3F\Lambda^3}{\delta(1-F^2)c^2}\frac{-\nu(u,v)}{r(u,v)^3}\int_{v_b(u)\le t\le s\le v}(s-t)g(t)g(s)\,\mathrm{d}t\,\mathrm{d}s.$$

The remaining integrations satisfy

$$\int_s^0\frac{-\nu(u,v)}{r(u,v)^3}\,\mathrm{d}v\le\frac{-\nu(u,s)}{c(2-\varepsilon)r_s^2}, \qquad (s-t)\int\frac{-\nu(u,s)}{r_s^2}\,\mathrm{d}u\le c^{-1}.$$

In the second integral one integrates only where both labels are in D, so $r_s\ge c(s-t)$; use $\mathrm{d}r_s=\nu(u,s)\,\mathrm{d}u$ and enlarge the nonnegative integral if its time set is disconnected. The pair contribution is therefore at most

$$\frac{3F\Lambda^3}{2\delta(1-F^2)(2-\varepsilon)c^4}J^2.$$

Here $\int_{t<s}g(t)g(s)\,\mathrm{d}t\,\mathrm{d}s=J^2/2$, since g is an L^1 density. Complex phases do not alter this argument.

For completeness, the three boundary contributions have the following bounds. The product of the boundary parts of θ/λ and $\mathcal{C}$ costs

$$C\int |\theta_b/\lambda_b|\,(\mu_b+|\theta_b/\lambda_b|\,|\xi_b/\nu_b|)\,\frac{\mathrm{d}\tau}{b}\le Ce_n.$$

For the product of $(\theta/\lambda)^j$ and the boundary curvature, the surviving kernel after the outer-radius integration is

$$C\int g(s)\,\mathrm{d}s\int b\,(\mu_b+|\theta_b/\lambda_b|\,|\xi_b/\nu_b|)\,\frac{-\nu(u,s)}{r_s^2}\,\mathrm{d}u.$$

Its mass part costs Ce_nJ. Its scalar part has the same bound by Cauchy–Schwarz and

$$\int\frac{b^3}{r_s^4}\left(\frac{\nu(u,s)}{\nu_b}\right)^2\mathrm{d}\tau\le b^{3-\varepsilon}\int_b^\infty r^{\varepsilon-4}\,\mathrm{d}r=\frac{1}{3-\varepsilon}.$$

Finally the product of $(\xi/\nu)^b$ and the derivative source in (2.111) costs at most

$$CF\int g(s)\,\mathrm{d}s\int\frac{|\xi_b/\nu_b|+C|\theta_b/\lambda_b|}{r_s}\,\mathrm{d}\tau\le CFJ\sqrt{e_n}.$$

The ratio retained in (2.108) is needed here. The final Cauchy–Schwarz kernel is $\int b\,\mathrm{d}\tau/r_s^2\le 1$, because $-\nu_b\le-\nu(u,s)$. For incoming connectors every estimate just used follows from Lemma 2.15, with complete norms listed in Lemma 2.14. Its constants depend only on the fixed ratio ρ and the late gap.

Thus summing the stages produces precisely the first four terms on the right side of (2.121).

Maxwell's equation gives $|Q_v|\le|q_0|M_0r\lambda F$. Substitution in (2.100), using (2.105), gives

$$\left|\frac{\lambda}{1-\mu}\mathcal{E}_j\right|\le Cq_0^2M_0^2F(-\nu)\lambda^2. \tag{2.124}$$

The required spacetime area bound has no time-duration factor:

$$\begin{aligned}\iint_D(-\nu)\lambda\,\mathrm{d}u\,\mathrm{d}v&=\int\mathrm{d}u\int_{b(u)}^{R(u)}(-\nu)(u,r)\,\mathrm{d}r\\&\le\int(-\nu)(u,0)R(u)\,\mathrm{d}u=\int_{R_{\mathrm{end}}}^{R_0}R\,\mathrm{d}R\le\frac12R_0^2.\end{aligned} \tag{2.125}$$

Here $-\nu(u,r)\le-\nu(u,0)$ follows from the nonnegative radial expansion coefficient. Since $\lambda\le\Lambda$, the same bound with an additional factor Λ controls the integral of $(-\nu)\lambda^2$. Thus (2.124) costs at most $C\zeta^2F$. The last term of (2.111), after multiplication by $(-\nu)\lambda^2|\theta/\lambda|/r^2$, costs at most $C\zeta^2F^3$.

For the piece containing $(\xi/\nu)^{\mathrm{em}}j$, substitute (2.110) in the curvature primitive. Its source is at most

$$Cq_0^2M_0^2F^2\frac{-\nu(u,v)}{r(u,v)^3}\int_{v_b(u)}^v r(u,s)^3g(s)\,\mathrm{d}s.$$

The outer v integral is bounded by a constant times $(-\nu(u,s))r(u,s)g(s)$, using the same radial growth estimate as in the neutral calculation. For each s,

$$\int(-\nu(u,s))r(u,s)\,\mathrm{d}u\le\frac12R_0^2.$$

This contribution is therefore at most $C\zeta^2F^2J$. Together with the unchanged neutral boundary and pair estimates, this proves (2.121).

The radial mass equation gives

$$m(u,r) \le m_b + \frac{1}{2}\int_b^r |\theta/\lambda|^2 \,\mathrm{d}x + Cq_0^2 M_0^2 F^2 r^3.$$

The last term has incoming optical cost bounded by $Cq_0^2M_0^2F^2\int(-\nu)r\,\mathrm{d}u \le C\zeta^2F^2$. For the scalar part, the boundary reconstruction has radial mass at most Cbe_n, since $2(1-F^2)>1$. Its incoming optical integral costs Ce_n. For the derivative-source part retain the finite upper endpoint:

$$\int_b^{r(u,v)} |(\theta/\lambda)^j|^2 \,\mathrm{d}x \le \frac{\Lambda}{c^2}\int_{v_b(u)\le s,t\le v} [v-\max\{s,t\}]g(s)g(t)\,\mathrm{d}s\,\mathrm{d}t.$$

This follows by discarding only the radial decay factors in (2.107) and using $\mathrm{d}x \le \Lambda\,\mathrm{d}v$. When both labels contribute, $r(u,v)\ge c[v-\min\{s,t\}]$, so

$$[v-\max\{s,t\}]\int \frac{-\nu(u,v)}{r(u,v)^2}\,\mathrm{d}u \le \frac{v-\max\{s,t\}}{c[v-\min\{s,t\}]} \le c^{-1}.$$

Tonelli bounds this optical contribution by $2\Lambda J^2/(\delta c^3)$. This proves (2.122) without a logarithmic integral at coincident labels. For the outgoing bound, use

$$\partial_r \log(-\nu) = \frac{\mu - Q^2/r^2}{(1-\mu)r} \le \frac{\mu}{(1-\mu)r}$$

and the same mass estimate. The outgoing scalar integral is bounded by expanding the square of (2.107); the pair of radial source points x,y has kernel

$$\int_{\max\{x,y\}}^R (x/r)^{1-F^2}(y/r)^{1-F^2}\frac{\mathrm{d}r}{r} \le \frac{1}{2(1-F^2)}\left(\frac{\min\{x,y\}}{\max\{x,y\}}\right)^{1-F^2} \le \frac{1}{2(1-F^2)}.$$

Converting the two source measures costs at most c^{-2} and gives $\int_b^R |\theta/\lambda|^2\,\mathrm{d}r/r \le C(|\theta_b/\lambda_b|^2 + J^2)$. □

2.6.4 Boundary forcing and connectors

On a cylinder $r=b$, use $D_\tau = (-\nu_b)^{-1}D_u + \lambda_b^{-1}D_v$ and the exact energy identity (2.49). The boundary forcing is estimated through the following derivative identity.

The chain rule for the boundary data is

$$D_\tau(\alpha/\lambda)_b = \frac{j_b}{(1-\mu_b)\lambda_b} + \frac{1}{b}\left[|\theta_b/\lambda_b|^2\frac{\theta_b}{\lambda_b} + \frac{\mu - Q^2/r^2}{1-\mu}\bigg|_b\frac{\theta_b}{\lambda_b} + \frac{iq_0Q\phi}{1-\mu}\bigg|_b\right]. \tag{2.126}$$

Variation on the complete boundary. Keep σ_n for the covariant variation on cylinder n alone. Let S be the covariant variation on the concatenation of cylinders and incoming connectors up to the current cutoff. Thus $\sum_n \sigma_n \le S$. At junctions the traces agree and unitary parallel transport concatenates, so no jump variation is added. On cylinders, (2.126), disjoint label intervals and $\mathrm{d}\tau = \lambda_b\,\mathrm{d}v_b$ bound the sum of variations by $c^{-1}J + C\sum e_n + C\zeta^2$. On connectors, (2.114), $B_n = \rho^n b_0$, and $e_n \le 1$ give

$$\sum_n \mathrm{TV}^D_{\text{connector n}}(\alpha/\lambda) \le C\sum_n e_n^{3/2} + Cq_0^2M_0^2\sum_n B_n^2\sqrt{e_n} \le C\sum_n e_n + C\zeta^2.$$

The stopped budgets have $e_n \le 1$ before the interaction estimate is applied. The full boundary variation consequently satisfies

$$S \le c^{-1}J + C\sum_n e_n + C\zeta^2. \tag{2.127}$$

The fixed factor $(1-\rho^2)^{-1}$ is included in C. We now combine (2.127) with the source estimate and the cooling recurrence; the finite-cutoff formulation below supplies the finiteness and continuity needed before absorption.

2.7 Shrinking cylinders, the complete cone, and continuation

We choose the cooling stages, close their five stopping bounds, and restart from the resulting complete cone.

2.7.1 Choice of scales and boundary budgets

Work after the late gap has been established. By taking the starting time later, the relative charge estimate gives everywhere in the late triangle

$$0 \le \mu - Q^2/r^2 \le \mu, \qquad 1-\mu \ge \delta > 0.$$

Choose a fixed $c > 0$ such that the adaptive exterior has $\lambda \ge 2c$ sufficiently late. There is also a fixed global upper bound $\lambda \le \Lambda$. Choose weak bounds $F < 1$ and $\varepsilon > 0$ satisfying $1 - F^2 > \max\{\varepsilon, 1/2\}$ and small enough for boundary observation. All constants depending on these fixed bounds are chosen before any shrinking scale.

Fix $0 < \rho < \chi < 1$. By Lemma 2.14, including the equivalence of Q_b/b and $m_b/b + |\theta_b/\lambda_b|^2$, choose a cooling fraction d with $C_{\rho,\delta}d < \chi$. Let H be the corresponding cooling-time constant and choose C_1 to absorb the squared-variation error after the connector. The choices below are made in the order

$$\varepsilon_{\mathrm{stop}}, \qquad \varepsilon_0, \qquad u_0 \uparrow 0.$$

On the late entry row put

$$R_0 = r(u_0, 0), \qquad b_0 = \ell(R_0), \qquad D_0 = \int_{v_{b_0}(u_0)}^{0} \left| \frac{\lambda j}{1-\mu}(u_0, v) \right| \mathrm{d}v.$$

The adaptive-exterior conclusions give

$$\frac{Q_{b_0}(u_0)}{b_0} \le \varepsilon_0, \qquad D_0 \to 0, \qquad \frac{b_0}{\varepsilon_0 R_0} \to 0 \quad (u_0 \uparrow 0), \tag{2.128}$$

where $\varepsilon_0 > 0$ is fixed before taking the limit. In particular it is an upper envelope, not the actual initial energy.

Until $R = r(u, 0)$ falls below $R_0/2$, the terminal estimate gives an amplitude bound

$$M_0 = C\left(1 + \sqrt{\log(R_i/R_0)}\right), \qquad M_0 \ge 1.$$

Choose the fixed constant large enough to include the exterior radial reconstruction under the stopping bounds below. Its precise value is independent of u_0. The charged error parameter is

$$\zeta^2 = q_0^2 M_0^2 R_0^2 \longrightarrow 0. \tag{2.129}$$

Set $b_n = \rho^n b_0$. At radius b_n, follow the timelike cylinder for clock length Hb_n/ε_n, where $\mathrm{d}\tau = (-\nu_b)\,\mathrm{d}u = \lambda_b\,\mathrm{d}v_b$. Let

$$\sigma_n = \int_{\text{cylinder } n} |D_\tau(\alpha/\lambda)|\,\mathrm{d}\tau.$$

Join its endpoint to radius b_{n+1} along the same incoming ray, and set

$$\varepsilon_{n+1} = \chi\varepsilon_n + C_1\sigma_n^2. \tag{2.130}$$

Proposition 2.11 gives the charged cooling inequality without an additive electromagnetic error. The connector estimate is multiplicative in the preceding energy budget. Therefore a fixed C_1 makes the next initial energy at most ε_{n+1}. Since $\varepsilon_n \geq \chi^n\varepsilon_0$, every prescribed stage has finite duration.

On a finite construction, including a partial current boundary piece, let S be its total covariant boundary variation, including connectors; then $\sum_n \sigma_n \leq S$. The integrated energy estimates give the budgets $e_n = C(\varepsilon_n + \sigma_n^2)$ used in the interaction proposition. Each connector uses its preceding budget. Iterating (2.130) and using $\sum \sigma_n^2 \leq S^2$ yields

$$\sum_n e_n \leq C(\varepsilon_0 + S^2), \qquad \sum_n \sqrt{e_n} \leq C(\sqrt{\varepsilon_0} + S). \tag{2.131}$$

These bounds hold for partial stages as well: intervals shorter than a full observation window are controlled directly by the energy identity and the endpoint-strip estimates.

2.7.2 The stopped exterior estimate

Use the incoming envelope g and its integral J defined below. Stop at the first failure of

$$\lambda \geq c, \quad R \geq R_0/2, \quad |\theta/\lambda| \leq F, \quad \mu/(1-\mu) \leq \varepsilon, \quad J + S \leq \varepsilon_{\text{stop}}. \tag{2.132}$$

These are exterior bounds. The previously proved gap and relative charge estimate hold throughout the original late triangle.

Finite cutoffs and continuity. The concatenated inner boundary is a continuous graph $v_b(u)$, increasing on cylinders and constant on connectors. For a proper cutoff t in a finite construction, define

$$D_t = \{u_0 \leq u \leq t,\ v_b(u) \leq v \leq 0\}, \qquad v_0 = v_b(u_0),$$

and, using the common representative of Lemma 2.1, set

$$g_t(v) = \left|\frac{\lambda j}{1-\mu}(u_0, v)\right| + \int_{u_0}^{t} \mathbf{1}_{\{v_b(s)\leq v\}} \left|D_u\left(\frac{\lambda j}{1-\mu}\right)(s, v)\right| \mathrm{d}s, \qquad J(t) = \int_{v_0}^{0} g_t(v)\,\mathrm{d}v.$$

Unitary transport makes g_t an envelope. On a fixed proper truncation, D_t is compact and has positive radius. The source in (2.99) is continuous in the primary fields, since Maxwell expresses Q_v in those fields. It is therefore bounded, independently of the rough initial L^1 density. In particular

$$J(t) = D_0 + \iint_{D_t} \left|D_u\left(\frac{\lambda j}{1-\mu}\right)\right| \mathrm{d}u\,\mathrm{d}v$$

is finite and continuous before absorption.

The boundary variation $S(t)$ is also finite and continuous: Lemma 2.2 gives an AC trace on each cylinder, and the incoming transport gives one on each connector. Their traces match at junctions, and parallel transport concatenates, so no atom is added. Finally D_t varies continuously in Hausdorff distance. Uniform continuity of the primary fields on a fixed compact truncation therefore makes all remaining stopping extrema, and $R(t)$, continuous.

Simultaneous initialization. With the preceding fixed parameters, take also $\varepsilon < 1/2$. Choose $\varepsilon_{\mathrm{stop}}$ small enough for the absorptions in (2.133) and for the stopped budgets to satisfy $e_n \le \min\{1, F^2\}$. Let C_* exceed the fixed constants in the closed variation, pointwise reconstruction and optical estimates. Choose $\varepsilon_0 > 0$ small, and then take the entry row sufficiently late, that

$$\begin{aligned} C_*(D_0 + \varepsilon_0 + \zeta^2) &< \tfrac{1}{2}\varepsilon_{\mathrm{stop}}, \\ C_*(\sqrt{\varepsilon_0} + D_0 + \varepsilon_0 + \zeta^2) &< F/2, \\ C_*(\varepsilon_0 + (D_0 + \varepsilon_0 + \zeta^2)^2 + \zeta^2) &< \min\left\{\frac{\varepsilon}{2(1+\varepsilon)}, \log(3/2)\right\}, \\ 2\left\{\frac{Hb_0}{\varepsilon_0(1-\rho/\chi)} + b_0\right\} &< R_0/4. \end{aligned}$$

These choices are possible because $D_0, \zeta^2 \to 0$ and $b_0/(\varepsilon_0 R_0) \to 0$ after ε_0 is fixed. The adaptive exterior entry estimates simultaneously give

$$\lambda(u_0, v) \ge 2c, \quad |\theta/\lambda|(u_0, v) < F/2, \quad \frac{\mu}{1-\mu}(u_0, v) < \varepsilon/2, \quad D_0 < \varepsilon_{\mathrm{stop}}/2$$

on that same entry row, with normalized observation energy at most ε_0, $R = R_0$, and $S = 0$. All five stops are therefore strict initially. The displayed margins are chosen for the simultaneous improvements proved next.

Before applying the interaction estimate, the recurrence and variation stop give

$$\sup_n \sqrt{e_n} \le C(\sqrt{\varepsilon_0} + \varepsilon_{\mathrm{stop}}) < F.$$

Thus (2.102), (2.104), and (2.105) apply on the stopped region. Their proofs used only the radial identities, the independent core gap, and the stopping bounds, so the amplitude and charge profiles are available before the interaction estimate is used.

The source estimate and the disjoint cylinder-label estimate give

$$\begin{aligned} &J \le D_0 + C(\varepsilon_0 + S^2) + CJ(\varepsilon_0 + S^2) + CFJ(\sqrt{\varepsilon_0} + S) + CFJ^2 + C\zeta^2(1 + J), \\ &S \le c^{-1}J + C(\varepsilon_0 + S^2) + C\zeta^2. \end{aligned} \tag{2.133}$$

The charged terms are summable by the boundary measure bound in Lemma 2.15 and the area bound (2.125); neither introduces a factor $1/\varepsilon_n$.

Choose $\varepsilon_{\mathrm{stop}}$ sufficiently small with all fixed constants retained. The second inequality in (2.133) absorbs CS^2 and gives $S \le C(J + \varepsilon_0 + \zeta^2)$. Under the stopping bound, $S^2 \le \varepsilon_{\mathrm{stop}} S$, so substitution in the first inequality permits the terms proportional to J to be absorbed. Choose ε_0 small and then u_0 late so that also ζ^2 is small. We obtain

$$J + S \le C(D_0 + \varepsilon_0 + \zeta^2). \tag{2.134}$$

The first initialization margin strictly improves the variation stop.

Radial reconstruction gives

$$|\theta/\lambda| \le |\theta_b/\lambda_b| + c^{-1}J, \qquad \mu(u, r) \le 2m_b/b + \sup_{b \le x \le r} |\theta/\lambda|^2 + C\zeta^2.$$

The budget sums and (2.134) thus strictly improve the two scalar/mass stopping bounds. The optical estimates give

$$\sup_v \int_{\text{exterior history of } v} \frac{(-\nu)\mu}{(1-\mu)r}\, \mathrm{d}u \le C(\varepsilon_0 + S^2 + J^2 + \zeta^2) < \log(3/2). \tag{2.135}$$

The logarithm of the blue-shift factor is nonnegative and no greater than this positive integral. Since every label starts with $\lambda \geq 2c$, (2.135) gives $\lambda \geq 4c/3$. The outgoing optical bound similarly gives $(-\nu(u,0))/(-\nu_b) \leq 2$ on the boundary.

The total cylinder clock is bounded by

$$\sum_n \frac{Hb_n}{\varepsilon_n} \leq \frac{Hb_0}{\varepsilon_0(1-\rho/\chi)}.$$

Connector clocks sum to b_0, because $\mathrm{d}\tau = -\,\mathrm{d}b$ on each incoming connector. Terminal radius loss is at most twice the total boundary clock. By (2.128), choose u_0 late enough that this loss is less than $R_0/4$. This strictly improves $R \geq R_0/2$.

2.7.3 Observation windows and existence of every stage

The charged observation condition involving the scalar background is satisfied uniformly: in a cylinder-parallel gauge its central value $H = (\alpha/\lambda)_b$ has $|H| \leq M_0 + F$, so $|q_0|b(1+|H|) \leq C\zeta$. This can be made smaller than the fixed observation threshold. No scalar translation is used in this assertion.

For a full boundary window, its earliest outgoing label u_- and latest incoming label $v_+ < 0$ determine an inward region contained in the proper original triangle $\{u_- \leq u \leq v \leq v_+\}$. Its gap is the independent late gap. The observation comparison derives its own normalized expansion bounds down to a fixed positive rescaled radius; it does not require the exterior lower bound c in this interior region. The comparison's characteristic margins therefore apply on every such window. Short and partial windows use only the energy identity and the integrated endpoint estimates already noted.

The preceding finite-cutoff argument proves continuity and finiteness before absorption, including at junctions. The simultaneous strict improvements therefore exclude a first failure on a proper finite stage. Such a stage cannot reach the axis at its positive final radius. It cannot meet $v = 0$, since its radius is at most $b_0 < R_0/2 \leq R$, and cannot reach $u = 0$ while $R \geq R_0/2$. It stays in a proper positive-radius region, where $\lambda > 0$ and $-\nu > 0$ continue its clock parametrization. Every prescribed finite stage consequently exists.

2.7.4 The complete cone and the local restart

The increasing boundary labels have limits u_*, v_*. Since $R \geq R_0/2$, one has $u_* < 0$; if $v_* = 0$, continuity would give $R(u_*) = \lim b_n = 0$, a contradiction. Thus the limit is proper. Since $b_n \to 0$ and radius is positive off the regular axis, $u_* = v_*$. The construction reaches an already existing axis point.

The budget sums give $\varepsilon_n, \sigma_n \to 0$, so $\theta_b/\lambda_b \to 0$ along the entire sequence of boundary pieces. Apply the row-variation estimate above on any compact subinterval of the limiting outgoing cone, pass to the limit, and then use its continuous axis trace. Lower semicontinuity of total variation gives

$$\mathrm{TV}_{[u_*,0]}\left(\frac{\alpha}{\lambda}(u_*,\cdot)\right) \leq C(D_0 + \varepsilon_0 + \zeta^2). \tag{2.136}$$

The local solution class gives AC of the datum on this proper complete cone, including its axis endpoint, whose scalar modulus is at most M_0.

Lemma 2.17 (Common null relabeling of the local solution class)**.** *Let T be increasing and C^1, with $T' > 0$ and T' locally BV. The common change $U = T(u)$, $V = T(v)$ preserves the regular-axis AC/BV class and its characteristic integral equations. It preserves the gauges $A_v = 0$ and $A_u|_\Gamma = 0$.*

Proof. All fields in the following formulas are composed with $(T^{-1}(U), T^{-1}(V))$. The scalars r, m, Q, ϕ, μ are unchanged, whereas

$$\widetilde{\lambda} = \frac{\lambda}{T'(v)}, \quad \widetilde{\nu} = \frac{\nu}{T'(u)}, \quad \widetilde{\theta} = \frac{\theta}{T'(v)}, \quad \widetilde{\xi} = \frac{\xi}{T'(u)}, \quad \widetilde{A}_U = \frac{A_u}{T'(u)}, \quad \widetilde{A}_V = 0, \quad \widetilde{\Omega}^2 = \frac{\Omega^2}{T'(u)T'(v)}.$$

In particular $\widetilde{\alpha}/\widetilde{\lambda} = \alpha/\lambda$ and $\widetilde{\beta}/\widetilde{\nu} = \beta/\nu$. On compact intervals T, T^{-1} are Lipschitz and their derivatives are positive, continuous, and BV. Characteristic change of variables preserves AC and normalized scalar variation; multiplication by $1/T'$ preserves the uniform BV expansion bounds. The displayed formulas also preserve continuity, signs, regular-axis traces and uniform vanishing of scalar variation on shrinking axis triangles.

For the integral equations substitute $\mathrm{d}U = T'(u)\,\mathrm{d}u$, $\mathrm{d}V = T'(v)\,\mathrm{d}v$. Covariant derivatives and expansions acquire exactly the displayed factors. No second derivative of T occurs: in $\partial_u\lambda$ the factor $T'(v)^{-1}$ is constant in the differentiated variable, and likewise for $\partial_v\nu$. The Maxwell and scalar equations transform identically. The class requires continuity of the potential coefficient, which is preserved; no incoming AC condition on that coefficient is needed. The common relabeling preserves the axis and its zero potential. □

Apply Lemma 2.17 with

$$T(t) = 2r(u_*, t), \qquad \widetilde{u} = T(u), \quad \widetilde{v} = T(v), \qquad u_* \le t \le 0.$$

Here $T'(t) = 2\lambda(u_*, t) > 0$ is continuous and BV, and $r(0, \widetilde{v}) = \widetilde{v}/2$ on the selected cone. Let $L = T(0) > 0$. Extend its normalized scalar datum constantly to $L + \Delta$, $0 < \Delta \le L$, and reconstruct the outgoing constraints from the regular axis. This preserves AC and variation, and the enlarged cone has radius at most $2R_0$. Equations (2.136) and (2.129) ensure the two smallness conditions of Theorem 3.2. It supplies a solution on the full enlarged closed triangle, with (L, L) strictly inside the new axis segment.

Extend the coordinate map affinely by

$$T(t) = L + T'(0)t \quad (t \ge 0), \qquad T(v_+) = L + \Delta.$$

It remains C^1 with positive locally BV derivative. Pulling back gives a solution in the same class and gauge on $\{u_* \le u \le v \le v_+\}$.

The full data agree on $u = u_*$: the radial average determines ϕ, the Maxwell and linear mass constraints determine Q, m from zero axis values, the expansion and potential constraints determine ν, A_u, and the outgoing wave transport determines β. Broad AC uniqueness on proper common triangles therefore identifies the developments wherever $u \ge u_*$ and $v < 0$. Continuity extends this equality to the old last cone except for the missing endpoint; uniqueness is never applied at that endpoint. Retaining the earlier old region gives the required solution on

$$\mathcal{Q} \cup \{u_* \le u \le v \le v_+\}, \qquad u_i < u_* < 0 < v_+.$$

On a characteristic crossing the gluing cone, matching traces allow the two integral equations and AC sections to concatenate, without an atom of variation. The same fact preserves BV expansions. Finite old/new covers give uniform class bounds on compact sets, including the short-axis variation property. Thus $(0, 0)$ is an interior point of the extended regular axis.

Lemma 2.18 (A positive incoming integral on a compact axis region)**.** *Suppose* $P = \{c \le u \le a,\ u \le v \le d\}$ *is a compact region of a solution in the regular-axis class and contains the indicated*

complete outgoing sections from the axis. Set $\lambda_- = \min_P \lambda > 0$, $\lambda_+ = \max_P \lambda$, $\Phi = \max_P |\phi|$, $R = \max_P r$ *and* $\delta = \min_P(1-\mu) > 0$. *For* $c \le w \le \min\{a, v\}$ *one has*

$$\int_c^w \frac{(-\nu)\mu}{(1-\mu)r}(u,v)\,\mathrm{d}u \le \log\frac{\lambda_+}{\lambda_-} + \frac{q_0^2\Phi^2R^2}{6\delta^2}. \tag{2.137}$$

When the upper endpoint is on the axis, the integral means its monotone improper limit.

Proof. For an off-axis cut integrate the exact radius equation:

$$\int_c^w \frac{(-\nu)\mu}{(1-\mu)r}\,\mathrm{d}u = \log\frac{\lambda(c,v)}{\lambda(w,v)} + \int_c^w \frac{(-\nu)Q^2}{(1-\mu)r^3}\,\mathrm{d}u.$$

Lemma 3.1, applied on each full outgoing section in P, gives

$$\frac{Q^2}{r^2} \le \frac{q_0^2\Phi^2r^2}{3\delta}\mu \le \frac{q_0^2\Phi^2r^2}{3\delta}.$$

Therefore

$$\int_c^w \frac{(-\nu)Q^2}{(1-\mu)r^3}\,\mathrm{d}u \le \frac{q_0^2\Phi^2}{3\delta^2}\int_c^w (-\nu)r\,\mathrm{d}u = \frac{q_0^2\Phi^2}{6\delta^2}\big(r(c,v)^2 - r(w,v)^2\big) \le \frac{q_0^2\Phi^2R^2}{6\delta^2}.$$

Monotone convergence gives the same bound at an axis endpoint, since the integrand on the left is nonnegative. □

The continuation contains the compact triangle $\{u_i \le u \le v \le 0\}$. Applying Lemma 2.18 proves (1.4) directly, with one constant for every incoming ray and every cut. On its compact terminal segment $[u_i, 0] \times \{0\}$, the solution-class bounds give BV for $\phi, \beta/\nu, \nu$. Since $-\nu$ is bounded away from zero, $\log(-\nu)$ is BV as well. This proves (1.3) and completes the extension argument.

3 Local theory for charged AC data of small total variation

This section supplies the finite-cone construction, characteristic variation estimates, strong AC limit, and broad local uniqueness used in Section 2. It follows Christodoulou's characteristic-variation philosophy but proves the charged estimates directly. The theorem concerns the AC subclass of small BV; it makes no assertion for arbitrary jump-BV data and uses no charged small-BV theorem as a black box.

3.1 Normalization and the local restart problem

The local problem needed below is finite in both null coordinates. It is not the asymptotically flat global small-data problem. In particular, one must retain the actual complex scalar value at the cone vertex: subtracting that value changes the charged equations.

We use

$$g = -\Omega^2\,du\,dv + r^2 g_{\mathbb{S}^2}, \qquad 1-\mu = -4\Omega^{-2}\nu\lambda, \qquad \mu = 2m/r, \quad \nu = r_u < 0 < r_v = \lambda.$$

Here m is the ordinary Hawking mass, not the charged renormalized mass $m + Q^2/(2r)$. The scalar normalization is $\phi = \sqrt{4\pi}\phi_{\text{Kommemi}}$. With the same q_0, Q, A, Kommemi's reduced equations [19,

Section 2.2.4, equations (18)–(27)] become, in the massless case,

$$
\begin{aligned}
r_{uv} &= \frac{\nu\lambda}{r}\frac{\mu - Q^2/r^2}{1-\mu}, \\
m_u &= \frac{(1-\mu)|\xi|^2}{2\nu} + \frac{Q^2\nu}{2r^2}, & m_v &= \frac{(1-\mu)|\theta|^2}{2\lambda} + \frac{Q^2\lambda}{2r^2}, \qquad (3.1)\\
Q_u &= q_0 r\,\mathrm{Im}(\bar\phi\xi), & Q_v &= -q_0 r\,\mathrm{Im}(\bar\phi\theta), \\
D_u\theta &= -\lambda\xi/r - \frac{iq_0Q\phi\nu\lambda}{(1-\mu)r}, & \partial_v\xi &= -\nu\theta/r + \frac{iq_0Q\phi\nu\lambda}{(1-\mu)r}. \qquad (3.2)
\end{aligned}
$$

The gauge is $A_v = 0$, $D_u = \partial_u + iq_0A_u$, with

$$
(A_u)_v = \frac{2\nu\lambda Q}{(1-\mu)r^2}, \qquad \theta = r\partial_v\phi, \qquad \xi = rD_u\phi, \qquad \alpha = \lambda\phi + \theta, \quad \beta = \nu\phi + \xi.
$$

After imposing $A_v = 0$, the remaining gauge transformation depends only on u; it may be chosen so that $A_u(u,u) = 0$. On a prescribed initial outgoing cone it changes α/λ by one constant unit complex number, and therefore preserves its total variation. Regularity at the center imposes $Q = 0$ there. The scaling $[q_0] = \text{length}^{-1}$ is also recorded in [19, Section 2.2.5].

Two useful consequences, obtained by direct substitution, are

$$
\begin{aligned}
D_u\left(\frac{\alpha}{\lambda}\right) &= \frac{-\nu}{r}\left[\frac{\mu - Q^2/r^2}{1-\mu}\frac{\theta}{\lambda} + \frac{iq_0Q\phi}{1-\mu}\right], \\
\partial_v\left(\frac{\beta}{\nu}\right) &= -\frac{\lambda}{r}\left[\frac{\mu - Q^2/r^2}{1-\mu}\frac{\xi}{\nu} - \frac{iq_0Q\phi}{1-\mu}\right], \qquad (3.3)\\
\partial_u \log\frac{\lambda}{1-\mu} &= \frac{|\xi|^2}{\nu r}, & \partial_v \log\frac{-\nu}{1-\mu} &= \frac{|\theta|^2}{\lambda r}. \qquad (3.4)
\end{aligned}
$$

The electromagnetic terms in the mass derivatives must be retained: ordinary Hawking mass has the required null monotonicities, whereas the charged renormalized mass has additional QQ_u/r and QQ_v/r terms.

Lemma 3.1 (Charge controlled relative to the Hawking mass)**.** *On an outgoing regular-center cone, assume $1-\mu > 0$. Then*

$$
Q^2(r) \le q_0^2\mu(r)r\int_0^r \frac{s^2|\phi(s)|^2}{1-\mu(s)}\,ds.
$$

In particular, if $1-\mu \ge \kappa > 0$ and $|\phi| \le M$, then

$$
\frac{Q^2}{r^2} \le \frac{q_0^2M^2r^2}{3\kappa}\,\mu. \qquad (3.5)
$$

Proof. The radial Maxwell equation and Cauchy–Schwarz give

$$
|Q(r)|^2 \le q_0^2\left(\int_0^r (1-\mu)|\theta/\lambda|^2\,ds\right)\left(\int_0^r \frac{s^2|\phi|^2}{1-\mu}\,ds\right).
$$

The radial mass constraint reads $2m_r = (1-\mu)|\theta/\lambda|^2 + Q^2/r^2$; its integral bounds the first factor by $2m(r) = \mu(r)r$. The last assertion follows by integrating s^2. □

Throughout the restart proof,

$$\alpha' = \lambda^{-1}\partial_v(\alpha/\lambda).$$

Thus the outgoing renormalized derivative used below is $\lambda(1-\mu)[\alpha' - |\theta/\lambda|^2(\theta/\lambda)/r]$. The density form of this expression is useful because its v integral is invariant under increasing relabelings of the outgoing coordinate.

The construction in the following sections is independent of a charged BV existence theorem. Its central estimates are proved for classical approximations, with constants depending only on the initial variation and the dimensionless coupling $|q_0|RM$. Strong comparison in fixed null labels then supplies the AC limit. This distinction matters: bounded variation alone would allow derivative concentration at the center, and weak compactness alone would not give the desired AC solution. Only the equations and normalization, not a local existence conclusion, are taken from [19].

3.2 Local well-posedness for AC data of small total variation

The remaining local problem can be solved by pairing the outgoing renormalized derivative with its incoming analogue. The characteristic estimates, stability argument, and construction in Sections 3.3–3.5 give an independent construction from the field equations. No charged existence or area theorem is used. The relation to Christodoulou's BV argument is the use of radial averaging and ordered null integrations; the complex curvature and the charge terms are treated explicitly here.

Theorem 3.2 (Small-BV local theory in the AC class)**.** *There are universal positive constants $\varepsilon_0, \delta_0, C$ with the following property. Prescribe $\alpha/\lambda \in \mathrm{AC}[0,T]$ on $u=0$, normalize $r(0,v) = v/2$, and impose the regular-center constraints, with zero central charge. Set*

$$R = T/2, \qquad \varepsilon = \mathrm{TV}_{[0,T]}(\alpha/\lambda)(0,\cdot), \qquad M = 1 + |\phi(0,0)| + 10\varepsilon.$$

If $\varepsilon \le \varepsilon_0$ and $|q_0|RM \le \delta_0$, the massless Einstein–Maxwell–charged-scalar-field equations have a regular-center solution on the full closed triangle

$$\mathcal{D}_T = \{0 \le u \le v \le T\}$$

in the gauge $A_v = 0$, $A_u(u,u) = 0$. The normalized scalar fields α/λ and β/ν are continuous and AC on their respective null sections; their axis values coincide with a common AC scalar trace. Moreover

$$\sup_u \mathrm{TV}(\alpha/\lambda)(u,\cdot) + \sup_v \int_0^v |D_u(\beta/\nu)(u,v)|\,du \le C\varepsilon, \tag{3.6}$$

$$\begin{aligned} |\phi| &\le M, \qquad |\theta/\lambda| + |\xi/\nu| \le C\varepsilon, \\ |\lambda - \tfrac{1}{2}| + |\nu + \tfrac{1}{2}| + \mu &\le C\varepsilon^2, \\ |Q| &\le C|q_0|M\varepsilon r^2. \end{aligned} \tag{3.7}$$

Both characteristic variations of $\lambda, \nu, \mu, \Omega^2$ are bounded by $C\varepsilon^2$. The geometric coefficients are AC on null sections, $r \in C^1$, and the equations hold in their integral and almost-everywhere senses. Ordinary incoming variation in the specified gauge is bounded by $C\varepsilon$ as well.

The solution map is Lipschitz in the fixed gauge and the norms specified in (3.62), with one universal constant for pairs of data satisfying the smallness assumptions. Writing Δ_0 for the initial $W^{1,1}$ distance specified there, the strong convergence of the characteristic scalar derivatives is quantified by

$$\sup_u \mathrm{TV}(\delta(\alpha/\lambda)(u,\cdot)) + \sup_v \mathrm{TV}(\delta(\beta/\nu)(\cdot,v)) \le C\Delta_0.$$

The incoming variation here is ordinary variation in the common fixed gauge. Derivative-envelope norms for the AC solutions are interpreted through the common representatives and integrable dominating envelopes constructed below. The geometric coefficients in the AC assertion are $\lambda, \nu, \mu, \Omega^2$, and $r \in C^1$ means C^1 on the closed triangle. Uniqueness holds on every closed subtriangle $\mathcal{D}_{T'}$, $0 < T' \le T$, against every solution in the stated local class with the same data and gauge; no smallness or derivative-envelope hypothesis is imposed on the competitor.

The smallness assumption is on total variation and on the charged length scale. There is no bound on the supremum or the L^2 norm of the initial derivative, and no Hölder or Dini hypothesis. A nonzero constant complex datum gives the flat zero-charge solution; it is retained throughout the argument. Arbitrary BV data with jumps require a separate measure-valued formulation of the axis trace and are not included in this theorem. The AC subclass of small BV is exactly the one used by the extension argument.

Organization of the proof. Section 3.3 supplies bounds for two renormalized characteristic derivatives and for the geometry. Section 3.4 proves strong stability, axis nonconcentration, and uniqueness in the larger local AC class. Section 3.5 constructs smooth solutions by a radial integral map, continues them across the full triangle, and passes to the AC limit. Together these arguments prove the theorem; none invokes a pre-existing charged low-regularity existence result. □

3.3 Independent characteristic variation estimates at a regular center

Here a *classical solution* has all nine primary fields $r, \lambda, \nu, m, Q, \phi, \theta, \xi, A_u$ jointly C^1 on the closed triangle, satisfies the reduced equations, and has a regular center. Mixed derivatives used below follow from the integral equations. The estimates depend only on initial total variation and the charged length scale; classical center orders justify boundary evaluations but do not enter their constants. The construction and passage to AC data are given in Section 3.5.

Work on $u_i \le u \le v \le V$, with $r(u_i, v) = (v - u_i)/2$, $A_v = 0$, and $A_u(u, u) = 0$. Write $R = (V - u_i)/2$ and let

$$\varepsilon = \mathrm{TV}\big((\alpha/\lambda)(u_i, \cdot)\big), \qquad M = 1 + |\phi(u_i, u_i)| + 10\varepsilon.$$

The small parameters are ε and $|q_0|RM$; the constant scalar background need not be small. Retain $\mathcal{C} = \mu + (1 - \mu)\theta\bar{\xi}/(\lambda\nu)$ and bootstrap

$$\frac{1}{4} \le \lambda, -\nu \le 1, \qquad 1 - \mu \ge \frac{1}{2}, \qquad |\phi| \le M, \qquad |\theta/\lambda|, |\xi/\nu| \le 4\varepsilon. \tag{3.8}$$

The constants 4 and 10 are fixed. We improve all inequalities simultaneously on complete triangles $\mathcal{D}_S = \{u_i \le u \le v \le S\}$, so every incoming integration to the axis remains in the constructed domain.

3.3.1 The two renormalized derivatives

Besides the already defined outgoing quantity j, introduce its incoming counterpart only for this proof:

$$j = (1 - \mu)\left[\partial_v\left(\frac{\alpha}{\lambda}\right) - \frac{\lambda}{r}\left|\frac{\theta}{\lambda}\right|^2\frac{\theta}{\lambda}\right], \tag{3.9}$$

$$k = (1 - \mu)\left[D_u\left(\frac{\beta}{\nu}\right) - \frac{\nu}{r}\left|\frac{\xi}{\nu}\right|^2\frac{\xi}{\nu}\right]. \tag{3.10}$$

Both have the same electromagnetic covariance as ϕ. Direct substitution of the field equations gives

$$D_u j = -\frac{|\xi|^2}{\nu r} j + \frac{\nu\theta}{r^2}(\mathcal{C} + 2\overline{\mathcal{C}}) + \mathcal{E}_j, \tag{3.11}$$

$$\partial_v k = -\frac{|\theta|^2}{\lambda r} k + \frac{\lambda\xi}{r^2}(\overline{\mathcal{C}} + 2\mathcal{C}) + \mathcal{E}_k, \tag{3.12}$$

$$\partial_v(r\mathcal{C}) = r\frac{\overline{\xi}}{\nu} j + \frac{\lambda Q^2}{r^2} - iq_0 Q\theta\overline{\phi}, \tag{3.13}$$

$$\partial_u(r\overline{\mathcal{C}}) = r\frac{\overline{\theta}}{\lambda} k + \frac{\nu Q^2}{r^2} + iq_0 Q\xi\overline{\phi}. \tag{3.14}$$

The exact sources are

$$\mathcal{E}_j = \frac{(-\nu)\lambda}{r^2}\left[5\frac{Q^2}{r^2}\frac{\theta}{\lambda} + iq_0 Q\left(3\frac{\theta}{\lambda} + \phi\right)\right] + \frac{iq_0(-\nu)Q_v\phi}{r}, \tag{3.15}$$

$$\mathcal{E}_k = \frac{(-\nu)\lambda}{r^2}\left[5\frac{Q^2}{r^2}\frac{\xi}{\nu} - iq_0 Q\left(3\frac{\xi}{\nu} + \phi\right)\right] + \frac{iq_0\lambda Q_u\phi}{r}. \tag{3.16}$$

One independent check on every sign is the interchange $u \leftrightarrow v$, $\lambda \leftrightarrow \nu$, $\theta \leftrightarrow \xi$, $Q \leftrightarrow -Q$; it takes (3.11), (3.13), and (3.15) exactly to their incoming counterparts. The formulation is covariant, so this check does not require the swapped gauge to retain $A_v = 0$.

Lemma 3.3 (The classical axis identity)**.** *In the gauge $A_v = 0$, $A_u|_\Gamma = 0$, a classical regular-center solution satisfies*

$$j(u,u) = k(u,u) = \frac{d}{du}\phi(u,u). \tag{3.17}$$

Proof. Write $x = v - u$. The outgoing average for ϕ and the continuity of λ and $\partial_v(\alpha/\lambda)$ give

$$\phi(u, u+x) = \frac{\alpha}{\lambda}(u,u) + \frac{x}{2}\partial_v\left(\frac{\alpha}{\lambda}\right)(u,u) + o(x).$$

In particular $\phi_v|_\Gamma = \frac{1}{2}\partial_v(\alpha/\lambda)|_\Gamma$. The regular-center orders $\theta/\lambda = O(r)$, $Q = O(r^3)$, $\mu = O(r^2)$ show from the normalized scalar transport equation that $\partial_u(\alpha/\lambda)|_\Gamma = 0$; the gauge term also vanishes there. Differentiating the common axis value $\phi(u,u) = (\alpha/\lambda)(u,u)$ therefore gives

$$(\phi|_\Gamma)' = \partial_v(\alpha/\lambda)|_\Gamma, \qquad \phi_u|_\Gamma = \phi_v|_\Gamma.$$

The incoming average, with its unitary parallel transport, gives $D_u(\beta/\nu)|_\Gamma = 2D_u\phi|_\Gamma$ in exactly the same way. Its endpoint parallel transport is the identity and $A_u|_\Gamma = 0$, so this last value is $(\phi|_\Gamma)'$. Both cubic corrections in the definitions of j, k are $O(r^2)$, and $1 - \mu = 1$ on the axis, proving the identity. □

For AC solutions (3.17) is an L^1 trace identity; its passage to the limit is proved in Section 3.5.3.

3.3.2 Radial Hardy estimates and the scalar reconstruction

At fixed u, $r\phi = \int_0^r \alpha/\lambda\, ds$. Therefore

$$\left|\frac{\theta}{\lambda}\right|(u,r) \le \frac{1}{r}\int_0^r s\,|\partial_s(\alpha/\lambda)|\,ds, \qquad \int_0^R \left|\frac{\theta}{\lambda}\right|\frac{dr}{r} \le \mathrm{TV}(\alpha/\lambda). \tag{3.18}$$

The identical incoming statement holds after parallel transport in the u direction. More precisely, integrate $D_u(r\phi) = \nu\beta/\nu$ from the axis and apply the unitary integrating factor. It gives

$$\int_{u_i}^{v} \frac{-\nu}{r} \left|\frac{\xi}{\nu}\right| du \le \int_{u_i}^{v} |D_u(\beta/\nu)|\, du. \tag{3.19}$$

These inequalities also hold on a shorter axis-to-point segment. The incoming variation in this section is always the covariant variation appearing on the right of (3.19).

Define nonnegative characteristic envelopes by

$$J(v) = \sup_{u_i \le u < v} |j(u,v)|, \qquad K(u) = \sup_{u < v \le V} |k(u,v)|.$$

Initially these are finite for a classical approximation. From (3.9), (3.18), and $|\theta/\lambda| \le 4\varepsilon$,

$$(1 - 16\varepsilon^2)\, \mathrm{TV}_{[u,v]}(\alpha/\lambda)(u,\cdot) \le \sup_{[u,v]}(1-\mu)^{-1} \int_u^v J(s)ds, \tag{3.20}$$

$$(1 - 16\varepsilon^2) \int_u^v |D_s(\beta/\nu)(s,v)|ds \le \sup_{[u,v]}(1-\mu)^{-1} \int_u^v K(s)ds. \tag{3.21}$$

Only one derivative is integrated, and it is integrated absolutely. No square-integrability of that derivative is used. In particular, the two scalar quadratic budgets are bounded by $C\varepsilon\|J\|_1$ and $C\varepsilon\|K\|_1$:

$$\sup_u \int_u^V \frac{\lambda}{r}|\theta/\lambda|^2 dv \le C\varepsilon\|J\|_1, \qquad \sup_v \int_{u_i}^v \frac{-\nu}{r}|\xi/\nu|^2 du \le C\varepsilon\|K\|_1. \tag{3.22}$$

3.3.3 Electromagnetic terms have a factor of the scalar variation

The outgoing Maxwell equation and the axis value $Q = 0$ imply

$$|Q| \le C|q_0|M\varepsilon r^2, \quad |Q_v| \le C|q_0|M\varepsilon\lambda r, \quad |Q_u| \le C|q_0|M\varepsilon(-\nu)r. \tag{3.23}$$

Consequently

$$|\mathcal{E}_j| + |\mathcal{E}_k| \le Cq_0^2M^2\varepsilon(-\nu)\lambda. \tag{3.24}$$

The charge contribution to either primitive for $\mathcal{C}$ obeys

$$\left|\frac{1}{r}\int_u^v \left(\frac{\lambda Q^2}{r^2} - iq_0Q\theta\overline{\phi}\right)(u,s)ds\right| \le Cq_0^2M^2\varepsilon^2r^2, \tag{3.25}$$

and similarly for the incoming primitive from (3.14). Since

$$\iint (-\nu)\lambda\, du\, dv \le CR^2,$$

the contribution of (3.24) to either envelope norm is at most $C(|q_0|RM)^2\varepsilon$. The curvature remainder (3.25) is smaller by two powers of ε after insertion in (3.11) or (3.12). In particular these estimates have no error independent of the initial variation. Constant scalar data have zero charge and give the flat constant-scalar solution, irrespective of the size of the constant.

3.3.4 The area estimate, with the integration order displayed

The homogeneous coefficient in (3.11) has nonnegative real part, but its integral is bounded by the second estimate in (3.22). Thus its amplification is at most $\exp(C\varepsilon\|K\|_1)$. In (3.12) the corresponding coefficient is nonpositive, so it can be discarded in an upper estimate for $|k|$.

The two oriented scalar curvature primitives are

$$\begin{aligned}\mathcal{C}^{v}_{\mathrm{sc}}(u,v) &= \frac{1}{r(u,v)}\int_u^v r(u,w)\frac{\overline{\xi}}{\nu}(u,w)j(u,w)\,\mathrm{d}w,\\ \overline{\mathcal{C}}^{u}_{\mathrm{sc}}(u,v) &= -\frac{1}{r(u,v)}\int_u^v r(s,v)\frac{\overline{\theta}}{\lambda}(s,v)k(s,v)\,\mathrm{d}s.\end{aligned} \tag{3.26}$$

The full $\mathcal{C}$ and $\overline{\mathcal{C}}$ are obtained by adding their respective electromagnetic remainders. These differ, so the displayed scalar primitives need not be complex conjugates.

Insert the outgoing primitive (3.13) into the curvature source of (3.11). The scalar part of its integrated absolute value is bounded by

$$\begin{aligned}&C\varepsilon\int_{u_i}^V du\int_u^V dv\,\frac{(-\nu)(u,v)\lambda(u,v)}{r(u,v)^3}\int_u^v r(u,a)|\xi/\nu|(u,a)J(a)\,da\\ &\le C\varepsilon\int_{u_i}^V J(a)\,da\int_{u_i}^a\frac{(-\nu)(u,a)}{r(u,a)}|\xi/\nu|(u,a)\,du\\ &\le C\varepsilon\left(\sup_a\int_{u_i}^a|D_u(\beta/\nu)(u,a)|du\right)\|J\|_1.\end{aligned} \tag{3.27}$$

In the second line the v integral is performed first:

$$\int_a^V\frac{\lambda(u,v)}{r(u,v)^3}dv\le\frac{1}{2r(u,a)^2},$$

and the bootstrap compares $(-\nu)(u,v)$ with $(-\nu)(u,a)$ by a fixed constant. The factor $r(u,a)$ already present in the primitive then leaves exactly the incoming Hardy integral in (3.19). This order of integration is essential. Replacing that Hardy integral by a supremum of ξ/ν would leave a divergent logarithm at the axis.

Use the incoming primitive (3.14) in the equation for k. The same calculation, with u and v interchanged, gives

$$\iint\frac{\lambda|\xi|}{r^2}|\overline{\mathcal{C}}^{u}_{\mathrm{sc}}|\,du\,dv\le C\varepsilon\left(\sup_u\mathrm{TV}(\alpha/\lambda)(u,\cdot)\right)\|K\|_1. \tag{3.28}$$

The curvature is complex throughout this calculation; the estimate $|\mathcal{C}+2\overline{\mathcal{C}}|\le 3|\mathcal{C}|$ is sufficient.

Initially, (3.18) gives $\|j(u_i,\cdot)\|_1\le\varepsilon+C\varepsilon^3$. Combining the preceding estimates gives

$$\|J\|_1\le\exp(C\varepsilon\|K\|_1)\left[\varepsilon+C\varepsilon^3+C\varepsilon\|K\|_1\|J\|_1+C(|q_0|RM)^2\varepsilon\right], \tag{3.29}$$

$$\|K\|_1\le\mathrm{TV}(\phi|_\Gamma)+C\varepsilon\|J\|_1\|K\|_1+C(|q_0|RM)^2\varepsilon. \tag{3.30}$$

To close the bootstrap, first exclude $\varepsilon=0$, for which the radial formulas give the constant-scalar flat solution. Fix C_* above the constants in the envelope, flux, mass and Raychaudhuri estimates. Choose the universal thresholds ε_0,δ_0 so that the following improvements hold for $\varepsilon\le\varepsilon_0$, $|q_0|RM\le\delta_0$, independently of the initial C^1 norm.

In addition to (3.8), bootstrap $\|J\|_1, \|K\|_1 \le 2\varepsilon$, where the envelopes and their integration intervals refer to $\mathcal{D}_S$. For a classical solution these assumptions hold on a sufficiently small initial triangle: at its vertex $\lambda = -\nu = 1/2$, $\mu = 0$, $\theta/\lambda = \xi/\nu = 0$, and $|\phi_{\text{vertex}}| < M$; the two envelope integrals tend to zero with its length. All quantities in this stopping argument depend continuously on S.

Under these bootstrap assumptions, (3.29)–(3.30) and Lemma 3.3 imply, first for J and then for K,

$$\|J\|_1 \le \varepsilon\{1 + C_*(\varepsilon^2 + (|q_0|RM)^2)\}, \qquad \|K\|_1 \le \varepsilon\{1 + C_*(\varepsilon^2 + (|q_0|RM)^2)\}.$$

Increasing C_* once covers the harmless exponential in the first inequality. Choose the thresholds so that each right-hand side is at most $9\varepsilon/8$. At this stage (3.20)–(3.21), using only $1 - \mu \ge 1/2$, give each characteristic scalar variation at most 3ε. The flux estimates are consequently $O(\varepsilon^2)$. The mass constraints and the Raychaudhuri integrations in the next paragraph, which use only these flux estimates and the provisional bootstrap, give

$$\mu \le C_*\varepsilon^2(1 + (|q_0|RM)^2), \qquad |\lambda - \tfrac{1}{2}| + |\nu + \tfrac{1}{2}| \le C_*\varepsilon^2(1 + (|q_0|RM)^2).$$

Thus the same thresholds can and do ensure

$$\frac{3}{8} \le \lambda, -\nu \le \frac{5}{8}, \qquad 1 - \mu \ge \frac{7}{8}. \tag{3.31}$$

Returning to (3.20)–(3.21) with this improved gap yields the explicit individual estimates

$$\begin{gathered} \|J\|_1 \le \frac{9}{8}\varepsilon, \qquad \|K\|_1 \le \frac{9}{8}\varepsilon, \\ \sup_u \mathrm{TV}(\alpha/\lambda)(u,\cdot) \le \frac{3}{2}\varepsilon, \qquad \sup_v \int_{u_i}^{v} |D_u(\beta/\nu)(u,v)|\, du \le \frac{3}{2}\varepsilon. \end{gathered} \tag{3.32}$$

Indeed $(8/7)(9/8)/(1-16\varepsilon^2) < 3/2$ for the chosen ε_0. The radial average formulas give $|\theta/\lambda|, |\xi/\nu| \le 3\varepsilon/2$. The axis identity bounds its scalar oscillation by $9\varepsilon/8$. An outgoing average now gives, pointwise,

$$|\phi(u,v)| \le |\phi_{\text{vertex}}| + \frac{9}{8}\varepsilon + \frac{3}{2}\varepsilon < 1 + |\phi_{\text{vertex}}| + 10\varepsilon = M.$$

Every bootstrap inequality is strictly improved, with the fixed M. Continuity in S closes the argument on every existing complete classical triangle. The two envelope norms therefore sum to at most $9\varepsilon/4$, and the two characteristic variations to at most 3ε.

3.3.5 Geometry, including variation in both null directions

The mass constraints, integrated from the axis, imply

$$\mu \le C\varepsilon^2(1 + q_0^2M^2r^2), \qquad \int \frac{\lambda\mu}{r}dv + \int \frac{(-\nu)\mu}{r}du \le C\varepsilon^2(1 + (|q_0|RM)^2).$$

The second statement means one estimate on each complete outgoing or incoming segment; it follows by radial Fubini from the corresponding quadratic scalar budgets and (3.23). Together with (3.22) and the two Raychaudhuri identities this gives

$$\lambda = \tfrac{1}{2} + O(\varepsilon^2), \qquad -\nu = \tfrac{1}{2} + O(\varepsilon^2), \qquad 1 - \mu = 1 + O(\varepsilon^2).$$

For $\lambda/(1-\mu)$ integrate the incoming Raychaudhuri equation from the initial cone. For $(-\nu)/(1-\mu)$ integrate the outgoing one from the axis, where $-\nu = \lambda$. This improves the geometric bootstrap without assuming a separate bound on the blue-shift factor.

For completeness, variation of λ in the v direction and variation of ν in the u direction require another cancellation. Introduce the two real expressions

$$\mathcal{L} = \partial_v \log\lambda + \frac{\lambda}{r}|\theta/\lambda|^2, \qquad \mathcal{N} = \partial_u \log(-\nu) + \frac{\nu}{r}|\xi/\nu|^2. \tag{3.33}$$

Direct differentiation gives the exact pair

$$\partial_u \mathcal{L} = \partial_v \mathcal{N} = -\frac{2\nu\lambda}{(1-\mu)r^2}\left(\operatorname{Re}\mathcal{C} - 2\frac{Q^2}{r^2}\right). \tag{3.34}$$

Here is an explicit algebraic check. Put temporarily $z = (\mu - Q^2/r^2)/(1-\mu)$. Differentiating $\partial_u \log\lambda = \nu z/r$ gives

$$\partial_u\partial_v \log\lambda = \frac{\nu\lambda}{r^2}\left[(1+z)|\theta/\lambda|^2 - 2z + \frac{2Q^2}{(1-\mu)r^2} - \frac{2QQ_v}{(1-\mu)\lambda r}\right].$$

Differentiating $\lambda|\theta/\lambda|^2/r$ gives

$$\frac{\nu\lambda}{r^2}\left[-(1+z)|\theta/\lambda|^2 - 2\operatorname{Re}\left(\frac{\overline{\theta}\xi}{\lambda\nu}\right) - 2\operatorname{Re}\left(\frac{iq_0Q\phi\overline{\theta}}{(1-\mu)\lambda}\right)\right].$$

The last terms cancel by $Q_v = -q_0 r\operatorname{Im}(\overline{\phi}\theta)$. The remaining terms are exactly (3.34). The temporary letter z is not needed elsewhere.

The same Fubini calculation as (3.27), now without the factor $|\theta/\lambda|$, shows

$$\iint \frac{(-\nu)\lambda}{r^2}|\mathcal{C}^v_{\mathrm{sc}}|du\,dv \le C\|K\|_1\|J\|_1 \le C\varepsilon^2.$$

The charge terms in (3.34) contribute at most $C(|q_0|RM)^2\varepsilon^2$. Since λ is constant initially, $\mathcal{L}(u_i, v) = \lambda|\theta/\lambda|^2/r$ has L^1 norm $O(\varepsilon^2)$. It follows that

$$\int \sup_u |\mathcal{L}(u,v)|dv \le C\varepsilon^2.$$

At the classical axis, $\mathcal{L}(u,u) = \mathcal{N}(u,u) = (\log\lambda|_\Gamma)'(u)$. The same transport estimate therefore bounds $\int \sup_v |\mathcal{N}(u,v)|du$ by $C\varepsilon^2$. Subtracting the quadratic corrections in (3.33) and using (3.22) gives both missing characteristic variation bounds. The cross variations follow directly from the equation for r_{uv}; the variation of μ follows directly from the mass constraints. Thus

$$\operatorname{TV}_{\mathrm{char}}\lambda + \operatorname{TV}_{\mathrm{char}}\nu + \operatorname{TV}_{\mathrm{char}}\mu + \operatorname{TV}_{\mathrm{char}}\Omega^2 \le C\varepsilon^2, \tag{3.35}$$

where each term means the supremum of its total variation over either family of null segments. The bound for Ω^2 follows from $\Omega^2 = -4\nu\lambda/(1-\mu)$. All denominators are bounded away from zero by the improved bootstrap. Maxwell directly gives $\operatorname{TV}_{\mathrm{char}} Q \le C|q_0|M\varepsilon R^2$. The potential is obtained from its outgoing integral with axis value zero. Its incoming absolute continuity requires all five terms in (3.64); their integrability and the vanishing moving-axis term are proved below. Its outgoing absolute continuity follows directly from the potential constraint. No electromagnetic gauge derivative is included in a scalar covariant variation. In the displayed axis gauge, the ordinary incoming variation is also finite: from $|A_u| \le C|q_0|M\varepsilon r$ and $|\beta/\nu| \le CM$,

$$\int |\partial_u(\beta/\nu)|du \le \int |D_u(\beta/\nu)|du + C(|q_0|RM)^2\varepsilon \le C\varepsilon.$$

The characteristic variation of the unnormalized quantities $\alpha = \lambda(\alpha/\lambda)$ and $\beta = \nu(\beta/\nu)$ follows by the product rule and is bounded by $C\varepsilon + CM\varepsilon^2$.

3.3.6 Use in the classical construction and the AC limit

The estimates above are a priori estimates on existing complete classical triangles. They yield the bounds needed by the construction, but do not by themselves establish continuation or compactness. Section 3.5.2 proves the uniform first-derivative estimates for each classical datum and the strip–corner continuation to the entire closed triangle.

For the AC passage, a uniform L^1 bound for J_n, K_n would not exclude concentration at the axis. We use the stronger comparison estimate (3.62): it makes the characteristic envelopes strongly Cauchy in L^1. Section 3.5.3 constructs their common representatives, proves uniform integrability and weighted flux nonconcentration, and passes each field equation and (3.17) to the limit. Thus the axis trace is AC and the geometry remains AC on null sections. Uniqueness in the larger local class is proved separately in Section 3.4.6, without assuming that a competitor has these envelopes.

3.4 Stability in fixed double-null coordinates and the AC limit

We prove strong $W^{1,1}$ stability for classical solutions, with constants depending only on the small-variation bounds. The resulting AC limit is constructed in Section 3.5.3; uniqueness against arbitrary solutions in the local AC/BV class is proved below.

Use the common coordinate domain

$$0 \le u \le v \le 2R, \qquad r(0,v) = v/2, \qquad r(u,u) = 0,$$

and impose $A_v = 0$, $A_u(u,u) = 0$. Fixed labels allow comparison of the initial L^1 densities without translating them. Assume

$$c_0 \le \lambda, -\nu \le C_0, \qquad 1-\mu \ge c_0, \qquad \left|\frac{\theta}{\lambda}\right| + \left|\frac{\xi}{\nu}\right| \le \varepsilon, \qquad |\phi| \le M, \quad M \ge 1. \tag{3.36}$$

For a pair of data sets, M is the larger amplitude parameter and ε a fixed universal multiple of their larger variation, bounding both scalar ratios and derivative envelopes. The theorem's thresholds allow this multiple. Set $\sigma = q_0^2 M^2 R^2$ and assume $\varepsilon^2 + \sigma$ small. Constants depend only on the geometric box.

In addition to the outgoing quantity j, define its incoming counterpart

$$k = (1-\mu)\left[D_u\left(\frac{\beta}{\nu}\right) - \frac{\nu}{r}\left|\frac{\xi}{\nu}\right|^2 \frac{\xi}{\nu}\right]. \tag{3.37}$$

For classical regular-center solutions their axis traces satisfy

$$k(u,u) = j(u,u) = \frac{\mathrm{d}}{\mathrm{d}u}\phi(u,u). \tag{3.38}$$

This is Lemma 3.3; its L^1 passage to the AC limit is justified in Section 3.5.3.

For a single solution put

$$J(v) = \sup_{0\le u\le v} |j(u,v)|, \qquad K(u) = \sup_{u\le v\le 2R} |k(u,v)|.$$

Assume the independent area argument has given

$$\|J\|_{L^1} + \|K\|_{L^1} \le C\varepsilon. \tag{3.39}$$

All subsequent estimates are independent of the initial derivative supremum.

3.4.1 The radial reconstruction and the correct Hardy exponent

The definitions give the exact equations

$$\partial_v\left(\frac{\theta}{\lambda}\right)+\frac{\lambda}{r}\left(1-\left|\frac{\theta}{\lambda}\right|^2\right)\frac{\theta}{\lambda}=\frac{j}{1-\mu}, \tag{3.40}$$

$$D_u\left(\frac{\xi}{\nu}\right)+\frac{\nu}{r}\left(1-\left|\frac{\xi}{\nu}\right|^2\right)\frac{\xi}{\nu}=\frac{k}{1-\mu}. \tag{3.41}$$

They are solved from the axis, where both ratios vanish. Parallel transport in the second equation is unitary. For any $p\le 1-\varepsilon^2$ we obtain

$$\left|\frac{\theta}{\lambda}(u,v)\right|\le C\int_u^v\left(\frac{r(u,s)}{r(u,v)}\right)^p J(s)\,\mathrm{d}s, \tag{3.42}$$

$$\left|\frac{\xi}{\nu}(u,v)\right|\le C\int_u^v\left(\frac{r(s,v)}{r(u,v)}\right)^p K(s)\,\mathrm{d}s. \tag{3.43}$$

The cubic correction changes the integrating factor. We use $p=3/4$ for individual fields and $p=1/2$ for differences.

Changing to radius and applying Fubini yields

$$\int_u^{2R}\frac{\lambda}{r}\left|\frac{\theta}{\lambda}\right|\,\mathrm{d}v\le C\|J\|_1, \tag{3.44}$$

$$\int_0^v\frac{-\nu}{r}\left|\frac{\xi}{\nu}\right|\,\mathrm{d}u\le C\|K\|_1. \tag{3.45}$$

Multiplication by ε gives the quadratic bounds. The incoming Hardy estimate concerns ξ/ν; it cannot be replaced by one for θ/λ.

3.4.2 The ordered kernel behind the comparison estimate

The scalar part of the exact curvature representation is

$$\mathcal{C}(u,v)=\frac{1}{r(u,v)}\int_u^v r(u,w)\frac{\overline{\xi}}{\nu}(u,w)j(u,w)\,\mathrm{d}w+\mathcal{C}_{\mathrm{em}}(u,v).$$

Insert (3.43) in the $\mathcal{C}$ term of the j transport equation. Since $r\asymp v-u$ and $r(u,w)\le C(v-u)$, its absolute value is bounded by

$$\frac{C\varepsilon}{(v-u)^{2+p}}\int_{u\le s\le w\le v}(w-s)^pK(s)J(w)\,\mathrm{d}s\,\mathrm{d}w. \tag{3.46}$$

After integrating in u and then in v, the only kernel integral is

$$\int_w^{2R}\frac{(w-s)^p}{(v-s)^{1+p}}\,\mathrm{d}v\le\frac{1}{p}.$$

Hence

$$\int_0^{2R}\int_0^v|\text{scalar curvature source of }j|\,\mathrm{d}u\,\mathrm{d}v\le C\varepsilon\|K\|_1\|J\|_1. \tag{3.47}$$

The same ordering proves the incoming estimate. It is precisely the factor $(w-s)^p$, expressing vanishing at the axis, that removes the logarithm. Replacing both characteristic ratios by their supremum before the integration loses this factor and gives an invalid estimate.

The complementary integration order, pulling out $\|J\|_\infty$ and integrating w first, gives

$$\int_s^v \frac{(w-s)^p}{(v-s)^{1+p}}\,\mathrm{d}w = \frac{1}{1+p}.$$

The resulting classical derivative bound and complete-triangle continuation are proved in Section 3.5.

3.4.3 Two solutions: the coupled comparison estimates

All comparisons below take place at the same (u, v) in the common axis gauge. Write δ for the difference and set

$$\Delta_j = \int_0^{2R} \sup_{u\le v} |\delta j(u,v)|\,dv, \qquad \Delta_k = \int_0^{2R} \sup_{v\ge u} |\delta k(u,v)|\,du,$$
$$G = \sup\left(|\delta\lambda| + |\delta\nu| + \frac{|\delta r|}{r_1}\right), \qquad \Delta_0 = \left|\delta\frac{\alpha}{\lambda}(0,0)\right| + \int_0^{2R}\left|\partial_v\delta\frac{\alpha}{\lambda}(0,v)\right|\,dv.$$

The quotient in G is understood by continuity at the axis. To expose the coupling, we temporarily name the norms which enter the proof:

$$P_v = \left\|\delta\frac{\theta}{\lambda}\right\|_\infty + \sup_u \int_u^{2R} \frac{\lambda_1}{r_1}\left|\delta\frac{\theta}{\lambda}\right|\,dv,$$
$$P_u = \left\|\delta\frac{\xi}{\nu}\right\|_\infty + \sup_v \int_0^{v} \frac{-\nu_1}{r_1}\left|\delta\frac{\xi}{\nu}\right|\,du,$$
$$\Delta_\phi = \|\delta\phi\|_\infty, \qquad \Delta_Q = \left\|\frac{\delta Q}{r_1^2}\right\|_\infty, \qquad \Delta_A = |q_0|R\|\delta A_u\|_\infty,$$
$$\Delta_\mu = \|\delta\mu\|_\infty + \sup_u \int_u^{2R} \frac{\lambda_1}{r_1}|\delta\mu|\,dv + \sup_v \int_0^{v} \frac{-\nu_1}{r_1}|\delta\mu|\,du.$$

The mass norm is scale invariant and includes both oriented Hardy integrals. An unweighted bound on δm would not suffice.

Before any coupled absorption, the estimates are

$$P_v \le C\Delta_j + C\varepsilon(G + \Delta_\mu), \tag{3.48}$$
$$P_u \le C\Delta_k + C\varepsilon(G + \Delta_\mu) + C\varepsilon\Delta_A, \tag{3.49}$$
$$\Delta_\phi \le C(\Delta_0 + \Delta_j + P_v + \varepsilon G), \tag{3.50}$$
$$\Delta_Q \le C|q_0|(MP_v + \varepsilon\Delta_\phi + M\varepsilon G), \tag{3.51}$$
$$\Delta_A \le C|q_0|R^2\Delta_Q + C\frac{\sigma\varepsilon}{M}(G + \Delta_\mu), \tag{3.52}$$
$$\Delta_\mu \le C\varepsilon(P_v + P_u) + C\varepsilon^2 G + C\varepsilon^2\Delta_\mu + C|q_0|M\varepsilon R^2\Delta_Q + C\sigma\varepsilon^2 G, \tag{3.53}$$
$$G \le C\Delta_\mu + C\varepsilon(P_v + P_u) + C\varepsilon^2 G. \tag{3.54}$$

We derive these inequalities before absorbing the coupled mass and geometric terms. In the radial scalar equations, the cubic map on the complex ball of radius ε has Lipschitz constant $3\varepsilon^2$. Use exponent $1/2$ for the difference equation and exponent $3/4$ for each undifferenced field. The error in λ/r is at most $CG\lambda_1/r_1$. Its convolution with the individual moment bound is harmless because, in the radius variable,

$$\int_s^v \left(\frac{r(t)}{r(v)}\right)^{1/2}\frac{dr(t)}{r(t)}\left(\frac{r(s)}{r(t)}\right)^{3/4} \le 4\left(\frac{r(s)}{r(v)}\right)^{1/2}.$$

The difference of the inverse gap multiplies j_2, respectively k_2, and costs $C\varepsilon\|\delta\mu\|_\infty$. An incoming parallel transport has modulus one, and the difference of two such transports has modulus at most $|q_0|\int|\delta A_u|\,du \le 2\Delta_A$. Radial Fubini proves the weighted parts of the first two lines. Only the small scalar cubic terms have been absorbed at this stage.

The axis identity gives $|\delta\phi(u,u)| \le \Delta_0 + \Delta_j$. Integration of $\phi_v = \lambda(\theta/\lambda)/r$ from the axis gives the third line. The charge and potential are reconstructed by

$$Q(u,v) = -q_0 \int_u^v \lambda r \ \mathrm{Im}\left(\overline{\phi}\frac{\theta}{\lambda}\right)(u,w)\,dw,$$
$$A_u(u,v) = \int_u^v \frac{2\nu\lambda Q}{(1-\mu)r^2}(u,w)\,dw.$$

Subtracting these integrals proves the fourth and fifth lines; in the latter use $|Q|/r^2 \le C|q_0|M\varepsilon$. The same calculation also proves the weighted charge estimate

$$\sup_u \int_u^{2R} \frac{\lambda_1}{r_1}\left|\delta\left(\frac{Q}{r^2}\right)\right| dv \le C|q_0|(MP_v + \varepsilon\Delta_\phi + M\varepsilon G). \tag{3.55}$$

Indeed, integrating $r^{-3}\int_0^r sf(s)\,ds$ in dr first leaves a constant times $\int|f(s)|\,ds/s$; the geometric difference is estimated in exactly this order.

For the mass estimate use, in either orientation,

$$\frac{dm}{dr} = \frac{1}{2}\left[(1-\mu)|\text{corresponding scalar ratio}|^2 + \frac{Q^2}{r^2}\right].$$

Integrate from the axis, divide by r, and subtract at the same characteristic labels. The difference of the scalar square is at most $C\varepsilon$ times the scalar difference; the difference of its gap factor is at most $C\varepsilon^2|\delta\mu|$. The geometric measure and averaging-radius differences cost $C\varepsilon^2 G$. Finally $|\delta(Q^2)| \le (|Q_1| + |Q_2|)|\delta Q|$ gives $C|q_0|M\varepsilon R^2\Delta_Q$ and $C\sigma\varepsilon^2 G$. Radial Fubini applies once again to each weighted integral of $|\delta\mu|/r$: for a nonnegative density f,

$$\int_0^L \frac{1}{r^2}\int_0^r f(s)\,ds\,dr \le \int_0^L \frac{f(s)}{s}\,ds.$$

This proves the sixth line, including its small Volterra term, uniformly down to the axis.

For the last line, subtract the two Raychaudhuri equations. The incoming quadratic flux difference is bounded by $C\varepsilon P_u + C\varepsilon^2 G$, and the outgoing one by $C\varepsilon P_v + C\varepsilon^2 G$. First integrate the equation for $\lambda/(1-\mu)$ from the initial cone. There $\lambda_i = 1/2$ for both solutions, so the difference of its initial value is bounded by $C\Delta_\mu$. Then integrate the equation for $(-\nu)/(1-\mu)$ from the axis, retaining the axis value of λ controlled by the first integration. The inverse gap costs $C\Delta_\mu$ and $r(u,v) = \int_u^v \lambda(u,w)\,dw$ controls $|\delta r|/r_1$. This proves (3.54).

Eliminate charge and potential, and combine the mass and geometric lines with the two scalar lines. Successive absorption gives

$$G + \Delta_\mu \le C\varepsilon(P_v + P_u) + C\frac{\sigma\varepsilon^2}{M}\Delta_\phi,$$
$$P_v + P_u \le C(\Delta_j + \Delta_k) + C\frac{\sigma\varepsilon^2}{M}\Delta_\phi.$$

Here the absorbed coefficients are $C(\varepsilon^2+\sigma)$. Insertion in the Δ_ϕ inequality and absorption of $C\sigma\varepsilon^2\Delta_\phi/M$ yield

$$\begin{aligned}\Delta_\phi &\le C(\Delta_0+\Delta_j+\Delta_k),\\ P_v+P_u &\le C(\Delta_j+\Delta_k)+C\sigma\varepsilon^2\Delta_0/M,\\ G+\Delta_\mu &\le C\varepsilon(\Delta_j+\Delta_k)+C\sigma\varepsilon^2\Delta_0/M.\end{aligned}\tag{3.56}$$

These estimates in particular imply the slightly coarser bound

$$G \le C\varepsilon(\Delta_j+\Delta_k)+C\sigma\Delta_0+C(\varepsilon^2+\sigma)G.\tag{3.57}$$

For the source calculation it is useful to retain the more informative charge and phase bounds

$$\Delta_Q \le C|q_0|M\left(\Delta_j+\Delta_k+\varepsilon G+\varepsilon\Delta_0/M\right),\tag{3.58}$$

$$\Delta_A \le C\frac{\sigma}{M}\left(\Delta_j+\Delta_k+\varepsilon G+\varepsilon\Delta_0/M\right).\tag{3.59}$$

3.4.4 Derivative differences: scalar kernels and charged sources

Use the two oriented scalar curvature primitives in (3.26), whose electromagnetic remainders differ.

Subtract the scalar moment representations inside these primitives, and then insert them into the differentiated scalar transport. All geometric factors are locally Lipschitz in the fixed box. The difference of their radial powers is estimated with exponent $1/2$, using the $3/4$ exponent of the undifferenced moments; this is the same convolution inequality displayed above. The inverse-gap differences cost Δ_μ. Differences of the nonlinear integrating factors cost $C\varepsilon(P_v+P_u)$, by the already proved Hardy bounds. For an incoming moment there is, additionally, a unitary phase difference bounded by $C\Delta_A$. Thus the positive ordered kernel of (3.46) gives, by replacing each of its factors in turn,

$$\begin{aligned}\iint |\delta(\text{scalar curvature source})|\,du\,dv &\le C\varepsilon^2(\Delta_j+\Delta_k)+C\varepsilon^3(G+\Delta_\mu)+C\varepsilon^3\Delta_A\\ &\le C\varepsilon^2(\Delta_j+\Delta_k)+C\varepsilon^3G+C\sigma\varepsilon^4\Delta_0/M.\end{aligned}$$

The phase contribution is $C\varepsilon\|J_2\|_1\|K_2\|_1\Delta_A$; (3.59) produces the displayed datum term (with the stronger denominator M^2 if retained). The kernel bound $1/p$, with $p=1/2$, is uniform at the axis.

For the electromagnetic sources, substitute Maxwell first:

$$\begin{aligned}\frac{iq_0(-\nu)Q_v\phi}{r} &= -iq_0^2(-\nu)\lambda\phi\,\mathrm{Im}\left(\overline{\phi}\frac{\theta}{\lambda}\right),\\ \frac{iq_0\lambda Q_u\phi}{r} &= -iq_0^2(-\nu)\lambda\phi\,\mathrm{Im}\left(\overline{\phi}\frac{\xi}{\nu}\right).\end{aligned}$$

In the following table, "ratio" is the scalar ratio in the relevant transport equation. Each monomial difference has area integral bounded by the indicated coefficient times $\Delta_j+\Delta_k+\varepsilon G+\varepsilon\Delta_0/M$:

monomial, with harmless fixed numerical factors suppressed	coefficient
$(-\nu)\lambda\,Q^2r^{-4}$ ratio	$C\sigma\varepsilon^2$
$iq_0(-\nu)\lambda\,Qr^{-2}$ ratio	$C\sigma\varepsilon$
$iq_0(-\nu)\lambda\,Qr^{-2}\phi$	$C\sigma$
$q_0^2(-\nu)\lambda\phi\,\mathrm{Im}(\overline{\phi}$ ratio$)$	$C\sigma$
electromagnetic curvature primitive in the scalar source	$C\sigma\varepsilon^2$

To check, write Q/r^2 in the first three rows and use $|Q|/r^2 \le C|q_0|M\varepsilon$, (3.58), and $\Delta_\phi \le C(\Delta_0 + \Delta_j + \Delta_k)$. The remaining coefficient R^2 is the area of the triangle. For the fourth row the difference of $\phi\,\mathrm{Im}(\bar\phi\,\mathrm{ratio})$ is bounded by $CM^2|\delta\mathrm{ratio}| + CM\varepsilon|\delta\phi|$. For the last row integrate the exact charged curvature terms: Q^2/r^2 and $q_0Q\theta\bar\phi$ (or $q_0Q\xi\bar\phi$) are $O(q_0^2M^2\varepsilon^2r^2)$ in their primitives. After division by the external radius this gives $\mathcal{C}_{\mathrm{em}} = O(q_0^2M^2\varepsilon^2r^2)$; its difference is estimated before the same integration. The external scalar ratio gives the fifth row. This accounts for all terms of $\mathcal{E}_j, \mathcal{E}_k$ and the charged curvature remainder.

The real homogeneous coefficient in the j transport has incoming integral $O(\varepsilon^2)$. Its difference, multiplied by j_2, has area norm at most $C\varepsilon^2P_u + C\varepsilon^3G$. There is also the covariant-derivative difference $-iq_0(\delta A_u)j_2$, whose area norm is at most $C\varepsilon\Delta_A$. For k the real homogeneous coefficient is nonpositive and its undifferenced propagator has norm at most one; its difference costs $C\varepsilon^2P_v + C\varepsilon^3G$. There is no additional v-transport gauge term, because $A_v = 0$. The initial radial constraint, the definition of j, and the cubic Hardy bound give $\|\delta j_i\|_1 \le C\Delta_0$. The incoming axis datum is $\delta k(u,u) = \delta j(u,u)$. Putting the preceding estimates into variation of constants yields

$$\Delta_j \le C\Delta_0 + C(\varepsilon^2+\sigma)(\Delta_j+\Delta_k) + C\varepsilon(\varepsilon^2+\sigma)G, \tag{3.60}$$

$$\Delta_k \le \Delta_j + C(\varepsilon^2+\sigma)(\Delta_j+\Delta_k) + C\varepsilon(\varepsilon^2+\sigma)G + C\sigma\varepsilon\Delta_0/M. \tag{3.61}$$

The first line absorbs the charged datum term into $C\Delta_0$. In the second it must be retained: its axis coefficient is exactly one, and changing the constant scalar background changes Maxwell forcing.

Insert (3.57) in these two inequalities and add the first inequality to one half of the second. Moving its axis term $\Delta_j/2$ to the left leaves positive coefficients for both derivative norms. For $C(\varepsilon^2+\sigma) < 1/4$ (after enlarging C once), all terms containing $\Delta_j + \Delta_k$ or G on the right are absorbed. Since $\alpha/\lambda = \phi + \theta/\lambda$ and $\beta/\nu = \phi + \xi/\nu$, this proves

$$\Delta_j + \Delta_k + G + \left\|\delta\frac{\alpha}{\lambda}\right\|_\infty + \left\|\delta\frac{\beta}{\nu}\right\|_\infty + \|\delta\phi\|_\infty \le C\Delta_0. \tag{3.62}$$

The constant is independent of the initial derivative supremum. The scalar–mass–charge–geometry system was closed before this final absorption.

The characteristic derivative convergence is ordinary, and not only covariant. Outgoing differentiation gives

$$\partial_v(\alpha/\lambda) = \frac{j}{1-\mu} + \frac{\lambda}{r}|\theta/\lambda|^2\theta/\lambda.$$

On incoming sections the exact identity is

$$\partial_u(\beta/\nu) = \frac{k}{1-\mu} + \frac{\nu}{r}|\xi/\nu|^2\xi/\nu - iq_0A_u\frac{\beta}{\nu}.$$

Their differences consequently satisfy

$$\sup_u \int |\partial_v\delta(\alpha/\lambda)|\,dv \le C(\Delta_j + \varepsilon\Delta_\mu + \varepsilon^2P_v + \varepsilon^3G),$$

$$\sup_v \int |\partial_u\delta(\beta/\nu)|\,du \le C\left(\Delta_k + \varepsilon\Delta_\mu + \varepsilon^2P_u + \varepsilon^3G + M\Delta_A + \frac{\sigma\varepsilon}{M}(\Delta_\phi + P_u)\right).$$

For the last two terms use $|\beta/\nu| \le CM$ and $|q_0|R\|A_u\|_\infty \le C\sigma\varepsilon/M$. Equations (3.56)– (3.62) therefore give the quantitative strong characteristic $W^{1,1}$ bound

$$\sup_u \mathrm{TV}\big(\delta(\alpha/\lambda)(u,\cdot)\big) + \sup_v \mathrm{TV}\big(\delta(\beta/\nu)(\cdot,v)\big) \le C\Delta_0. \tag{3.63}$$

3.4.5 Strong convergence of the geometric derivatives

To preserve AC geometry, compare the renormalized derivatives $\mathcal{L}, \mathcal{N}$ of (3.33), with transport equation (3.34) and common axis value $\mathcal{N}(u,u) = \mathcal{L}(u,u)$.

Subtract their source terms. The same ordered kernel (3.46), with the prefactor ε omitted, yields

$$\iint \left| \delta \left[\frac{2\nu\lambda}{(1-\mu)r^2} \left(\operatorname{Re} \mathcal{C} - 2\frac{Q^2}{r^2} \right) \right] \right| \mathrm{d}u \, \mathrm{d}v \\ \leq C\varepsilon(\Delta_j + \Delta_k) + C\varepsilon^2 G + C\sigma(\Delta_0 + \Delta_j + \Delta_k + \varepsilon G).$$

On the initial cone, $\mathcal{L} = \lambda|\theta/\lambda|^2/r$; the difference of these initial values has L^1 norm at most $C\varepsilon\Delta_0$ by the Hardy estimate. Integrate first $\mathcal{L}$ from this cone and then $\mathcal{N}$ from its common axis trace. Together with (3.62) this proves

$$\int \sup_u |\delta\mathcal{L}(u,v)| \, \mathrm{d}v + \int \sup_v |\delta\mathcal{N}(u,v)| \, \mathrm{d}u \leq C(\varepsilon + \sigma)\Delta_0.$$

Subtracting their quadratic corrections, whose differences have already been estimated in the weighted flux norm, gives strong L^1 convergence of $\partial_v\lambda$ and $\partial_u\nu$ uniformly on characteristic sections. The equation for r_{uv} gives the cross derivatives: its difference is controlled by the oriented mass estimate and the incoming and outgoing Hardy bounds. The mass constraints similarly give strong L^1 convergence of both characteristic derivatives of μ. For example

$$\partial_v\mu = \frac{\lambda}{r}\left[(1-\mu)\left|\frac{\theta}{\lambda}\right|^2 + \frac{Q^2}{r^2} - \mu \right],$$

and the radial mass average controls its last term with exactly the same Hardy integral. Therefore λ, ν, μ and $\Omega^2 = -4\nu\lambda/(1-\mu)$ have AC characteristic limits. The Maxwell constraints provide the corresponding assertion for Q.

Incoming absolute continuity of the potential. This requires differentiating the whole radial integral, including its moving lower endpoint. For classical solutions let

$$T(u,w) = \frac{2\nu\lambda Q}{(1-\mu)r^2}(u,w).$$

Leibniz' rule gives exactly

$$\partial_u A_u(u,v) = -T(u,u) \\ + \int_u^v \left[\frac{2\nu\lambda Q_u}{(1-\mu)r^2} + \frac{2\lambda Q\nu_u}{(1-\mu)r^2} + \frac{2\nu Q\lambda_u}{(1-\mu)r^2} + \frac{2\nu\lambda Q\mu_u}{(1-\mu)^2 r^2} - \frac{4\nu^2\lambda Q}{(1-\mu)r^3} \right](u,w)\, dw. \tag{3.64}$$

The endpoint term vanishes by Maxwell reconstruction and the scalar moment estimate:

$$\frac{|Q(u,v)|}{r(u,v)^2} \leq C|q_0|M \sup_{u\leq w\leq v} |\theta/\lambda(u,w)| \longrightarrow 0 \quad (v-u \longrightarrow 0).$$

The limit is uniform for uniformly integrable J envelopes; the weaker bound $Q = O(r^2)$ would not suffice.

We verify integrability of each of the five displayed terms on $0 \le u \le w \le v_*$, uniformly in $v_* \le 2R$. For the first term use $Q_u = q_0 \nu r \operatorname{Im}(\bar\phi\, \xi/\nu)$ and the incoming moment inequality:

$$\int_0^{v_*} dw \int_0^w \frac{|\xi/\nu(u,w)|}{w-u}\, du \le CR\|K\|_1.$$

For example inserting the moment and interchanging $u \le s \le w$ reduces the inner integration to $\int_0^s (w-s)^p (w-u)^{-1-p} du \le 1/p$. Thus this contribution is at most $C|q_0|MR\|K\|_1$.

For the second term use the already proved geometric renormalization

$$\nu_u = \nu\mathcal{N} - \frac{\nu^2}{r}|\xi/\nu|^2, \qquad \int \sup_v |\mathcal{N}(u,v)|\, du \le C\varepsilon^2.$$

With $|Q|/r^2 \le C|q_0|M\varepsilon$, its integral is bounded by

$$C|q_0|M\varepsilon R\left(\|\mathcal{N}\|_{L^1_u L^\infty_v} + \varepsilon\|K\|_1\right).$$

For the third and fourth terms, substitute the exact equations

$$\lambda_u = \frac{\nu\lambda}{(1-\mu)r}\left(\mu - \frac{Q^2}{r^2}\right), \qquad \mu_u = \frac{\nu}{r}\left[(1-\mu)|\xi/\nu|^2 + \frac{Q^2}{r^2} - \mu\right].$$

The incoming mass and scalar Hardy integrals and $Q^2/r^2 \le Cq_0^2M^2\varepsilon^2 r^2$ bound their sum by $C|q_0|MR\varepsilon^3(1+\sigma)$. Finally the charge constraint, with radial Fubini, gives

$$\int_u^{v_*} \frac{\lambda|Q|}{r^3}(u,w)\, dw \le C|q_0|M\int_u^{v_*} \frac{\lambda}{r}|\theta/\lambda|(u,w)\, dw.$$

Integration in u bounds the fifth term by $C|q_0|MR\|J\|_1$. We have proved

$$\sup_{v_*\le 2R}\int_0^{v_*} |\partial_u A_u(u,v_*)|\, du \le C|q_0|MR\varepsilon. \tag{3.65}$$

This controls the incoming section integral, without a pointwise bound on $\partial_u A_u$.

There is a corresponding strong difference estimate. Put $\Delta_{\mathcal{N}} = \int \sup_v |\delta\mathcal{N}(u,v)|\, du$; the preceding geometric comparison has already shown $\Delta_{\mathcal{N}} \le C(\varepsilon+\sigma)\Delta_0$. Subtract the five terms of (3.64) before integrating. The difference of the first is bounded using P_u, Δ_ϕ, G and Δ_μ. In the second use

$$\delta\nu_u = \nu_1\delta\mathcal{N} + \delta\nu\,\mathcal{N}_2 - \delta\left(\frac{\nu^2}{r}|\xi/\nu|^2\right);$$

the coefficient of $\delta\mathcal{N}$ is bounded by $C|q_0|M\varepsilon$, while terms with $\delta(Q/r^2)$ use Δ_Q and the integrable undifferenced $\mathcal{N}_2$. For the third and fourth terms use the two exact equations above; their differences are controlled by the weighted parts of Δ_μ, P_u, and the charge bounds. For the fifth term use (3.55). Keeping a convenient common upper bound gives

$$\begin{aligned}\sup_v \int_0^v |\partial_u \delta A_u(u,v)|\, du &\le C|q_0|MR\left[P_v + P_u + \varepsilon\left(\frac{\Delta_\phi}{M} + G + \Delta_\mu + \Delta_{\mathcal{N}}\right)\right] + CR\varepsilon^2\Delta_Q \\ &\le C|q_0|MR(1+\varepsilon+\sigma)\Delta_0. \end{aligned}\tag{3.66}$$

The term $CR\varepsilon^2\Delta_Q$ includes differences of Q/r^2 against undifferenced geometric derivatives; their electric parts carry an additional factor σ. The outgoing derivative follows directly from the potential constraint and the same comparison norms.

For the constructed AC solutions, (3.66) makes the classical incoming derivatives strongly Cauchy, uniformly over sections. Uniform convergence of A_u identifies the limit with its distributional derivative, including the axis endpoint. Section 3.5.3 carries out this passage, together with the other equations, after complete-triangle classical existence has been established.

3.4.6 Uniqueness in the full local AC/BV class

For gluing, a competing solution need not possess derivative envelopes. We use the undifferentiated equations and ordinary characteristic BV bounds in the common gauge $A_v = 0$, $A_u|_\Gamma = 0$.

Lemma 3.4 (Uniqueness on a small axis triangle)**.** *Consider two solutions in the stated local AC/BV class on $\mathcal{D}_{a,b} = \{a \le u \le v \le b\}$. Suppose they have the same outgoing data and coordinates on $u = a$, and that both obey fixed positive upper and lower bounds for $\lambda, -\nu$ and $1 - \mu$. Suppose, for both solutions, that*

$$|\phi| \le M, \quad M \ge 1, \qquad \left|\frac{\theta}{\lambda}\right|, \left|\frac{\xi}{\nu}\right| \le \varepsilon \le 1,$$

that the oscillation of each normalized scalar field on this triangle is at most ε, and that its ordinary variation on every characteristic segment is at most L. Let R be an upper bound for both area radii and put $\sigma = q_0^2 M^2 R^2$. There is a constant depending only on the geometric bounds such that the two solutions agree if

$$C\{\varepsilon(L+1) + \sigma\}\{1 + C\varepsilon(L+1)\} < 1.$$

No derivative envelope is assumed for either solution.

Proof. Constants depend only on the geometric box. Work first off the axis; the integrability estimates below justify passage to it. The radii are comparable by their expansion bounds, and the radial mass constraint with zero axis mass gives $0 \le \mu < 1$.

Write δ for the first solution minus the second and use the continuous-field distances

$$D = \left\|\delta\frac{\alpha}{\lambda}\right\|_\infty + \left\|\delta\frac{\beta}{\nu}\right\|_\infty, \qquad G = \|\delta\lambda\|_\infty + \|\delta\nu\|_\infty + \left\|\frac{\delta r}{r_1}\right\|_\infty.$$

The last quotient is understood by its continuous diagonal extension. Radial averaging gives the exact cancellation

$$\begin{aligned}\delta\phi(u,v) = {} & \frac{1}{r_1(u,v)}\int_u^v \lambda_1(u,s)\delta\frac{\alpha}{\lambda}(u,s)\,\mathrm{d}s \\ & + \frac{1}{r_1(u,v)}\int_u^v \delta\lambda(u,s)\left(\frac{\alpha_2}{\lambda_2}(u,s) - \phi_2(u,v)\right)\mathrm{d}s.\end{aligned}$$

Consequently,

$$\|\delta\phi\|_\infty + \left\|\delta\frac{\theta}{\lambda}\right\|_\infty + \left\|\delta\frac{\xi}{\nu}\right\|_\infty \le C(D + \varepsilon G). \tag{3.67}$$

The geometric error carries the scalar oscillation ε.

Ordinary variation, covariant variation, and Hardy's inequality. Maxwell reconstruction from the regular axis, followed by the outgoing potential constraint, gives

$$|Q| \le C|q_0|M\varepsilon r^2, \qquad |A_u| \le C|q_0|M\varepsilon r. \tag{3.68}$$

Indeed $Q_v = -q_0 r\lambda\,\mathrm{Im}(\overline{\phi}\,\theta/\lambda)$, and $(A_u)_v = 2\nu\lambda Q/((1-\mu)r^2)$. Thus, on each incoming segment,

$$\mathrm{TV}^D(\beta/\nu) \le \mathrm{TV}(\beta/\nu) + |q_0|\int |A_u|\,|\beta/\nu|\,\mathrm{d}u \le L + C\sigma\varepsilon. \tag{3.69}$$

In the incoming frame parallel transported from the axis, and with outward-increasing radius, $\partial_r(r\phi) = \beta/\nu$. Thus

$$\left(\frac{\xi}{\nu}\right)(r) = \frac{1}{r}\int_{[0,r]} s\;\mathrm{d}(\beta/\nu)(s).$$

Here the right side uses the BV derivative of the transported field. Tonelli gives

$$\int_0^R \frac{1}{r}\left|\frac{1}{r}\int_{[0,r]} s\,\mathrm{d}(\beta/\nu)(s)\right|\mathrm{d}r \le \int_{[0,R]}\left(\int_s^R \frac{s}{r^2}\,\mathrm{d}r\right)|\,\mathrm{d}(\beta/\nu)|(s) \le \mathrm{TV}^D(\beta/\nu).$$

There is no endpoint atom because the trace is continuous. Together with the outgoing version in $A_v = 0$, this proves

$$\begin{aligned}\sup_v \int_a^v \frac{-\nu}{r}\left|\frac{\xi}{\nu}\right|\mathrm{d}u &\le L + C\sigma\varepsilon,\\ \sup_u \int_u^b \frac{\lambda}{r}\left|\frac{\theta}{\lambda}\right|\mathrm{d}v &\le L.\end{aligned} \tag{3.70}$$

Undifferenced mass budgets. The two exact oriented representations are

$$2m(u,v) = \int_u^v (-\nu)\left[(1-\mu)\left|\frac{\xi}{\nu}\right|^2 + \frac{Q^2}{r^2}\right](s,v)\,\mathrm{d}s, \tag{3.71}$$

$$2m(u,v) = \int_u^v \lambda\left[(1-\mu)\left|\frac{\theta}{\lambda}\right|^2 + \frac{Q^2}{r^2}\right](u,s)\,\mathrm{d}s. \tag{3.72}$$

For either orientation the scalar part of the integrated mass ratio is bounded by

$$\int_0^R \frac{1}{r^2}\int_0^r |\text{corresponding scalar ratio}(s)|^2\,\mathrm{d}s\,\mathrm{d}r \le \varepsilon\int_0^R |\text{corresponding scalar ratio}(s)|\frac{\mathrm{d}s}{s}.$$

The electric part is bounded by $Cq_0^2M^2\varepsilon^2R^2$ using (3.68). Thus, for each solution,

$$\sup_v \int_a^v \frac{(-\nu)\mu}{r}\,\mathrm{d}u + \sup_u \int_u^b \frac{\lambda\mu}{r}\,\mathrm{d}v \le C\varepsilon L + C\sigma\varepsilon^2. \tag{3.73}$$

The gap bound permits an additional factor $(1-\mu)^{-1}$. These budgets control terms in which only the scalar ratio is subtracted.

Differences of charge and mass. Subtracting the outgoing Maxwell integral and using (3.67) gives

$$\frac{|\delta Q|}{r_1^2} \le C|q_0|M(D+\varepsilon G).$$

Using $\mathrm{Im}(\overline{\phi}\,\theta/\lambda)$ in the Maxwell integrand also gives

$$\left|\delta\left(\frac{Q^2}{r^2}\right)\right| \le Cq_0^2M^2\varepsilon r_1^2(D+\varepsilon G).$$

Subtracting either oriented mass representation first gives

$$\|\delta\mu\|_\infty \le C\varepsilon(D+\varepsilon G) + C\varepsilon^2\|\delta\mu\|_\infty + C\sigma\varepsilon(D+\varepsilon G).$$

Absorb the middle term after choosing ε small. For the integrated differences, apply the same-orientation Fubini calculation before estimating the remaining scalar ratio. The kernel is

$$\int_s^R r^{-2}\,\mathrm{d}r \le s^{-1}.$$

The scalar-square difference is bounded by $(|\xi_1/\nu_1| + |\xi_2/\nu_2|)|\delta(\xi/\nu)|$ in the incoming direction, and by the corresponding outgoing expression in the other direction. Equation (3.70) controls the first factor after integration. Differences of the radial kernels cost CG times the undifferenced quadratic budget. A difference of $1-\mu$ inside the representation contributes at most $C\varepsilon^2$ times the same integrated mass-difference norm, and is absorbed. The electric term is integrated using $\int_0^R r\,\mathrm{d}r = R^2/2$. We obtain

$$\|\delta\mu\|_\infty \le C\varepsilon D + C\varepsilon^2 G + C\sigma\varepsilon(D+\varepsilon G), \tag{3.74}$$

$$\sup_v \int_a^v \frac{-\nu_1}{r_1}|\delta\mu|\,\mathrm{d}u + \sup_u \int_u^b \frac{\lambda_1}{r_1}|\delta\mu|\,\mathrm{d}v \le C(L+\sigma\varepsilon)(D+\varepsilon G). \tag{3.75}$$

The potential constraint now yields

$$|\delta A_u(u,v)| \le C|q_0|Mr_1(u,v)(D+\varepsilon G). \tag{3.76}$$

Indeed, subtract $\nu, \lambda, Q, (1-\mu)^{-1}, r^{-2}$ in its radial integral. The inverse-gap and inverse-radius differences are bounded by $C|\delta\mu|$ and CG/r_1^2; the row length is comparable to r_1.

Both optical integrations, including the axis term. Integrate the first equation in (3.4) from the common outgoing initial cone:

$$\log\frac{\lambda}{1-\mu}(u,v) = \log\frac{\lambda}{1-\mu}(a,v) - \int_a^u \frac{-\nu}{r}\left|\frac{\xi}{\nu}\right|^2(s,v)\,\mathrm{d}s.$$

The initial difference vanishes. Equations (3.67) and (3.70) bound the integral difference by $C(L+\sigma\varepsilon)(D+\varepsilon G)$. Together with (3.74), this controls $\delta\lambda$, including its axis value.

The second optical integration is

$$\log\frac{-\nu}{1-\mu}(u,v) = \log\lambda(u,u) + \int_u^v \frac{\lambda}{r}\left|\frac{\theta}{\lambda}\right|^2(u,s)\,\mathrm{d}s.$$

Retain the axis-lapse difference, bounded by the first integration. The outgoing Hardy estimate controls the integral, and $\delta r = \int_u^v \delta\lambda\,\mathrm{d}s$ gives

$$G \le C(L+1)D + C\varepsilon(L+1)G \qquad (\varepsilon,\sigma\le 1). \tag{3.77}$$

Choosing $C\varepsilon(L+1)\le 1/2$ gives $G \le C(L+1)D$.

Closing the scalar comparison. Subtract the first transport equation in (3.3), including the gauge term $-iq_0A_u(\alpha/\lambda)$ when using ordinary derivatives. The difference of the geometric source has three types of terms: a ratio difference times an undifferenced mass budget; a factor bounded by ε times the integrated mass difference; and a geometric difference times an undifferenced budget and a scalar ratio. These are bounded, respectively, by (3.73), (3.75), and both estimates together. The Maxwell and gauge differences cost $C\sigma(D+\varepsilon G)$ by (3.68) and (3.76). The common outgoing datum therefore gives

$$\left\|\delta\frac{\alpha}{\lambda}\right\|_\infty \le C(\varepsilon L+\sigma)(D+\varepsilon G).$$

Integrate the second equation in (3.3) outwards from the axis. Its initial difference equals $\delta(\alpha/\lambda)(u,u)$ and is kept. The same estimates, now in the outgoing orientation, give

$$\left\|\delta\frac{\beta}{\nu}\right\|_\infty \le \left\|\delta\frac{\alpha}{\lambda}\right\|_\infty + C(\varepsilon L+\sigma)(D+\varepsilon G).$$

Thus

$$D \le C(\varepsilon L + \sigma)\{1 + C\varepsilon(L+1)\}D.$$

The stated smallness implies $D = 0$ and then $G = 0$. Radial averaging determines ϕ and then θ, ξ. Maxwell, mass and potential reconstruction determine Q, m, A_u, and $\Omega^2 = -4\nu\lambda/(1-\mu)$ also agrees. □

Lemma 3.5 (Positive-radius rectangle uniqueness)**.** *Two continuous integral solutions in the stated class on a closed characteristic rectangle, with the same data on its two past sides, agree if r, λ, $-\nu$ and $1-\mu$ have positive lower bounds there.*

Proof. Use the nine primary fields $r, \lambda, \nu, m, Q, \phi, \theta, \xi, A_u$. Integrate the equations for $r, \lambda, m, Q, \phi, \theta$ in u from the left side; here $\phi_u = \xi/r - iq_0 A_u \phi$ and $\lambda_u = r_{uv}$. Integrate the equations for ν, ξ, A_u in v from the bottom side. On the stated bounded, nondegenerate set, all right-hand sides are locally Lipschitz functions of these nine fields.

Let C_* bound these Lipschitz constants for both solutions on the rectangle. On a cell whose entering sides already agree, subtraction of the integral equations gives

$$\sup_{\text{cell}} |\delta(r, \lambda, \nu, m, Q, \phi, \theta, \xi, A_u)| \le C_*(\Delta u + \Delta v) \sup_{\text{cell}} |\delta(r, \lambda, \nu, m, Q, \phi, \theta, \xi, A_u)|.$$

Choose a finite grid with $C_*(\Delta u + \Delta v) < 1/2$ and process cells by increasing sum of their indices. Each cell has common entering sides from the boundary data or preceding cells, so contraction proves equality on the whole rectangle. □

Proposition 3.6 (Uniqueness on complete and apex-omitted triangles)**.** *Two solutions in the local AC/BV class with the same characteristic datum, initial radius gauge and regular-axis gauge agree on every common closed characteristic triangle. The same holds when only its terminal axis apex is omitted.*

Proof. The outgoing constraints determine all primary fields on the initial cone from its normalized scalar datum and radius. For ϕ, Q, m, ν, A_u, use (3.78)–(3.82). The coefficient of the mass Volterra equation is integrable by the outgoing BV Hardy estimate. For ξ, integrate its outgoing transport from the axis value zero; the same Hardy estimate makes its scalar source integrable. Thus the initial full characteristic fields agree.

Translate labels to $\mathcal{D}_T = \{0 \le u \le v \le T\}$. Fix L, M and a geometric box on a compact neighborhood of $(0,0)$. Continuity makes the scalar ratios and normalized-field oscillations small on a shrinking axis triangle; its shrinking radius makes $q_0^2 M^2 R^2$ small. Lemma 3.4 starts equality on $\mathcal{D}_s$ for some $s > 0$.

Let s_* be the supremum of the s for which equality holds on $\mathcal{D}_s$. If $s_* < T$, continuity gives equality on the closed triangle $\mathcal{D}_{s_*}$. Choose $\eta > 0$ small and put $a = s_* - \eta > 0$. On a fixed compact neighborhood of (s_*, s_*) the two solutions have finite common bounds L, M and a common geometric box. Choose η and then $\delta > 0$ so that the axis triangle

$$\{a \le u \le v \le s_* + \delta\}$$

lies in that neighborhood and satisfies the smallness conditions of Lemma 3.4.

First consider the positive-radius strip

$$[0, a] \times [s_*, s_* + \delta].$$

Its entering sides agree by the initial data and equality on $\mathcal{D}_{s_*}$. Since $v - u \ge \eta$, rectangle uniqueness applies. The strip and old triangle therefore provide common data on the entire base $u = a$,

$a \le v \le s_* + \delta$. The axis lemma proves equality on the new triangle. Together the old triangle, strip and new axis triangle contain $\mathcal{D}_{s_*+\delta}$, contradicting the definition of s_*. Therefore $s_* = T$, and continuity includes the terminal edge and apex.

If the terminal apex is absent, perform this argument on each $\mathcal{D}_{T'}$ with $T' < T$. This gives equality where $v < T$. At a point of the remaining edge $v = T$, $u < T$, equality follows by continuity from $v < T$. No uniqueness assertion at the absent apex is used. □

3.5 Construction in fixed double-null coordinates

We now construct the solution from the small-variation estimates. Fix

$$\mathcal{D}_T = \{0 \le u \le v \le T\}, \qquad r(0,v) = v/2, \qquad A_v = 0, \qquad A_u(u,u) = 0.$$

For the datum $(\alpha/\lambda)_i \in \mathrm{AC}[0,T]$, retain

$$\varepsilon = \mathrm{TV}\left(\left(\frac{\alpha}{\lambda}\right)_i\right), \qquad M = 1 + |\phi(0,0)| + 10\varepsilon, \qquad R = T/2.$$

The amplitude parameter M is the fixed number in the theorem; only ε and $|q_0|RM$ are required to be small.

3.5.1 A direct local construction for C^1 data

For a candidate pair $(\frac{\alpha}{\lambda}, \lambda)$ with $\lambda > 0$, reconstruct every other coefficient on each outgoing section by

$$r(u,v) = \int_u^v \lambda(u,s)\,ds, \qquad \phi(u,v) = \frac{1}{r(u,v)}\int_u^v \lambda(u,s)\frac{\alpha}{\lambda}(u,s)\,ds, \tag{3.78}$$

$$Q(u,v) = -q_0 \int_u^v r\lambda\ \mathrm{Im}(\overline{\phi}\frac{\alpha}{\lambda})(u,s)\,ds. \tag{3.79}$$

The axis value of ϕ is $\frac{\alpha}{\lambda}(u,u)$. The outgoing mass constraint is linear in m. Its explicit solution is

$$m(u,v) = \frac{1}{2}\int_u^v \lambda\left(|\frac{\theta}{\lambda}|^2 + \frac{Q^2}{r^2}\right)(u,s)\exp\left[-\int_s^v \frac{\lambda|\frac{\theta}{\lambda}|^2}{r}(u,t)\,dt\right]ds. \tag{3.80}$$

Set $\mu = 2m/r$, and then define

$$\nu(u,v) = -\lambda(u,u)\exp\left[\int_u^v \frac{\lambda}{r}\frac{\mu - Q^2/r^2}{1-\mu}(u,s)\,ds\right], \tag{3.81}$$

$$A_u(u,v) = \int_u^v \frac{2\nu\lambda Q}{(1-\mu)r^2}(u,s)\,ds. \tag{3.82}$$

The remaining two transport equations define the Picard map:

$$\lambda_{\mathrm{new}}(u,v) = \frac{1}{2}\exp\left[\int_0^u \frac{\nu}{r}\frac{\mu - Q^2/r^2}{1-\mu}(s,v)\,ds\right], \tag{3.83}$$

$$\begin{aligned}\left(\frac{\alpha}{\lambda}\right)_{\mathrm{new}}(u,v) &= \exp\left[-iq_0\int_0^u A_u(s,v)\,ds\right]\left(\frac{\alpha}{\lambda}\right)_i(v)\\ &\quad + \int_0^u \exp\left[-iq_0\int_s^u A_u(t,v)\,dt\right]\frac{-\nu}{r}\left[\frac{\mu - Q^2/r^2}{1-\mu}(\frac{\theta}{\lambda}) + \frac{iq_0Q\phi}{1-\mu}\right](s,v)\,ds.\end{aligned} \tag{3.84}$$

All quantities on the right are reconstructed from the old pair. The same map starts from any later outgoing section, replacing the factor $1/2$ by its prescribed positive λ datum.

Lemma 3.7 (A closed nondegenerate ball for the radial map)**.** *For C^1 data satisfying Theorem 3.2, the radial map has a unique fixed point on $0 \le u \le a$, $u \le v \le T$, for some $a > 0$ depending on the initial C^1 norm. Later-cone data in a bounded C^1 set with the strict a priori margins admit one common existence time.*

Proof. Only in this proof abbreviate $h = \alpha/\lambda$. The case of constant data has already been solved explicitly. Otherwise put

$$\|(h,\lambda)\|_{1,T} = \sup(|h| + |\lambda|) + T\sup(|h_v| + |\lambda_v|), \qquad 0 < \rho \le \min(1/16, \varepsilon/8).$$

The scale T stays fixed even for shorter later cones. On the closed truncated strip use the ball of radius ρ about $(h_i(v), 1/2)$ in the space of jointly continuous $h, \lambda, h_v, \lambda_v$, with prescribed trace on $u = 0$. It has $1/4 \le \lambda \le 3/4$ and, on every outgoing row,

$$\mathrm{TV}\, h \le \varepsilon + \rho \le 2\varepsilon, \qquad |\phi| \le |h_i(0)| + \varepsilon + \rho < M, \qquad |h - \phi| \le 2\varepsilon.$$

Write temporarily $V = 2\varepsilon$. Positivity of the weights in the average, followed by the explicit charge and mass formulas, gives, before any division by $1 - \mu$,

$$|Q(r)| \le \frac{|q_0|MV}{2} r^2, \qquad 0 \le \mu(r) \le \frac{1}{r}\int_0^r \left(V^2 + \frac{q_0^2 M^2 V^2 s^2}{4}\right) ds = V^2\left(1 + \frac{q_0^2 M^2 r^2}{12}\right).$$

Here the mass exponential is at most one and $r \le 3T/4$. The universal smallness thresholds therefore ensure $\mu \le 1/8$ on the entire ball, before the inverse-gap formulas for ν, A_u and the transport coefficients are used.

Put $x = v - u$. Let B control the ball's C^1_v norm, amplitude, T and the fixed positive inverse bounds; C_B may also depend on q_0. The radial formulas give

$$r \asymp x, \qquad h - \phi = O_B(x), \qquad Q = O_B(x^3), \qquad m = O_B(x^3),$$
$$\mu = O_B(x^2), \qquad \nu + \lambda(u,u) = O_B(x^2), \qquad A_u = O_B(x^2).$$

Their differentiated forms are

$$\phi_v = \frac{\lambda}{r}(h - \phi), \qquad Q_v = -q_0 r \lambda\, \mathrm{Im}(\bar\phi h),$$
$$m_v = \frac{\lambda}{2}\left((1-\mu)|h-\phi|^2 + Q^2/r^2\right), \qquad \mu_v = 2m_v/r - \lambda\mu/r,$$
$$\nu_v = \frac{\nu\lambda}{r}\frac{\mu - Q^2/r^2}{1-\mu}, \qquad (A_u)_v = \frac{2\nu\lambda Q}{(1-\mu)r^2}.$$

Their orders are, respectively, $O_B(1), O_B(x^2), O_B(x^2), O_B(x), O_B(x), O_B(x)$. For example $\mathrm{Im}(\bar\phi h) = \mathrm{Im}(\bar\phi(h-\phi))$, which gives the additional power in Q_v. The two ordinary transport right-hand sides are

$$a = \frac{\nu}{r}\frac{\mu - Q^2/r^2}{1-\mu}, \qquad b = -iq_0 A_u h - \frac{\nu}{r}\left[\frac{\mu - Q^2/r^2}{1-\mu}(h-\phi) + \frac{iq_0 Q\phi}{1-\mu}\right].$$

Here $a = O_B(x)$, $a_v = O_B(1)$, $b = O_B(x^2)$, and $b_v = O_B(x)$. These assertions require only h_v, λ_v, not a second derivative of the datum.

For two pairs at distance d in $\|\cdot\|_{1,T}$, the differences of

$$\phi,\ h-\phi,\ Q,\ m,\ \mu,\ \nu + \lambda(u,u),\ A_u,\ a,\ b$$

are bounded by $C_B d$ times, respectively,

$$1, x, x^3, x^3, x^2, x^2, x^2, x, x^2;$$

the first v derivatives have the same bounds with one power of x removed, except that the derivative of ϕ is bounded by $C_B d$. The full ν difference is $O_B(d)$, including its axis value. To verify these weighted bounds, use $|\delta r| \le x\|\delta\lambda\|_\infty$, and write

$$h - \phi = \frac{1}{r}\int_u^v \lambda(u,s)[h(u,v) - h(u,s)]\,ds.$$

The bracket is bounded by $x\|h_v\|_\infty$; its difference is bounded by xd/T. This proves the weighted difference for $h - \phi$ and, using the first displayed derivative identity, for ϕ_v and $(h-\phi)_v$. Next use $|\delta(r^{-q})| \le C_B x^{-q}\|\delta\lambda\|_\infty$ and the charge integral to obtain the Q, Q_v differences. The mass integral and its exponential give the m, m_v, μ, μ_v differences in the stated order. At this point

$$\left|\delta\frac{1}{1-\mu}\right| \le (8/7)^2|\delta\mu|,$$

so the last two radial integrals and the formulas for a, b give the remaining bounds, including the diagonal value $\lambda(u,u)$ and the first v derivative of the inverse gap. The same difference-quotient formulas give their continuous center limits.

Let $\mathcal{P}$ denote the two displayed Picard formulas. Differentiating those formulas once in v and using the preceding bounds gives

$$\|\mathcal{P}(h,\lambda) - (h_i, 1/2)\|_{1,T} \le aC_B, \qquad \|\mathcal{P}(h_1,\lambda_1) - \mathcal{P}(h_2,\lambda_2)\|_{1,T} \le aC_B d.$$

The unitary phase in the formula for h has the same bounds; its v derivative is the integral of $(A_u)_v$. Choose $a \le \min(T, \rho/(2C_B), 1/(2C_B))$. The closed ball is preserved and the contraction factor is at most $1/2$. Banach iteration gives the fixed point. Its transport equations supply continuous u derivatives; the recovery below gives all nine primary fields in C^1, including ξ.

For a later cone $u = u_0$, replace the reference pair by $(h_i(v), \lambda_i(v))$ and replace $1/2$ in the λ map by $\lambda_i(v)$. In its application here, $3/8 \le \lambda_i \le 5/8$, $\mathrm{TV}\, h_i \le 3\varepsilon/2$, and the scalar bound has the strict margin just proved. A ball of radius at most $\min(1/16, \varepsilon/8)$, reduced if necessary, has $\mathrm{TV}\, h \le 2\varepsilon$, positive lower λ, and the same mass bound. The preceding proof applies without change. On a bounded C^1 set, the constants and ball radius are uniform because the norm uses the original T and introduces no inverse length of the new cone. Hence a common $\tau > 0$ constructs a later cone of length L up to $u - u_0 = \min(L, \tau)$, including its entire closed triangle when $L < \tau$. The same time works for small perturbations preserving the strict margins. □

At a fixed point, (3.81) and (3.83) imply

$$\partial_u r = -\lambda(u,u) + \int_u^v \partial_u\lambda(u,s)\,ds = \nu(u,v),$$

so the reconstructed radial function has the prescribed null derivatives. Set $\theta = r\partial_v\phi$ and $\xi = rD_u\phi$. The equation for $\frac{\alpha}{\lambda}$ gives

$$D_u\partial_v(r\phi) = \frac{\nu\lambda}{(1-\mu)r^2}\left(\mu - \frac{Q^2}{r^2} - iq_0Q\right)r\phi.$$

Together with (3.82), this is precisely the charged wave equation, including both of its first-order transport identities.

The fixed point gives $h = \alpha/\lambda, \lambda$ jointly C^1, and the radial formulas give $r, \phi, m, Q, \nu, A_u, \theta$ in C^1. To prove the same for ξ without assuming a second u derivative of ϕ, commute D_u and ∂_v in the displayed wave equation and integrate outwards:

$$\beta(u,v) = -\lambda(u,u)h(u,u) + \int_u^v \frac{\nu\lambda}{(1-\mu)r}\left(\mu - \frac{Q^2}{r^2} + iq_0Q\right)\phi(u,s)\,ds.$$

The sign $+iq_0Q$ uses $(A_u)_v = 2\nu\lambda Q/((1-\mu)r^2)$. Direct differentiation of the reconstruction integrals, before using the incoming constraints, gives

$$\mu_u = O_B(r), \quad Q_u = O_B(r^2), \quad h_u = O_B(r^2), \quad \lambda_u = O_B(r), \quad \phi_u = O_B(1).$$

Thus the integrand is $O_B(r)$ and its u derivative has a continuous center limit. The formula defines a jointly C^1 field β. The already continuous $D_u(r\phi)$ satisfies the same outgoing integral equation and axis datum; their equality follows by ordinary integral uniqueness. Consequently $\xi = \beta - \nu\phi$ is C^1.

For completeness, the two unused incoming constraints propagate without an additional field equation. Define their defects by

$$Z = \partial_u Q - q_0 r\,\mathrm{Im}(\bar{\phi}\xi), \qquad E = \partial_u m - \frac{(1-\mu)|\xi|^2}{2\nu} - \frac{Q^2\nu}{2r^2}.$$

Differentiating the outgoing constraints and using the charged wave equations gives the triangular system

$$\partial_v Z = 0, \qquad \partial_v E = -\frac{|\theta|^2}{\lambda r}E + \frac{\lambda Q}{r^2}Z.$$

The regular-axis defects vanish, so first $Z = 0$ and then $E = 0$. With $\Omega^2 = -4\nu\lambda/(1-\mu)$, the fixed point therefore solves the full reduced system.

3.5.2 Classical continuation to the closed triangle

Work on existing complete subtriangles $\mathcal{D}_S = \{0 \le u \le v \le S\}$, where the closed bootstrap (3.31) and (3.32) apply. We must bound first derivatives for each classical datum; these bounds may depend on its C^1 norm. Retain the opposite radial moment from the incoming reconstruction, with a fixed admissible exponent $p \in (0,1)$:

$$\left|\frac{\xi}{\nu}(u,t)\right| \le C\int_u^t \left(\frac{t-a}{t-u}\right)^p K(a)\,da, \qquad 0 < p < 1. \tag{3.85}$$

The outgoing analogue uses J. The loss from exponent 1 to p absorbs the cubic correction.

Insert (3.85) into the radial formula for $\mathcal{C}$, and bound only j by $\|J\|_\infty$ and $|\theta/\lambda|$ by $C\varepsilon$. After integration of the j equation along an incoming ray, the resulting term is at most

$$C\varepsilon\|J\|_\infty \int_0^v K(a)(v-a)^{p+1}\int_0^{\min(u,a)}(v-s)^{-p-2}\,ds\,da \le C\varepsilon\|K\|_1\|J\|_\infty.$$

The kernel remains bounded when $a \to v$. The homogeneous exponent is $O(\varepsilon\|K\|_1)$, and the electromagnetic source contributes $Cq_0^2M^2T\varepsilon$. The reflected calculation, with the axis datum $k = j$ from Lemma 3.3, gives

$$\|J\|_\infty \le C\|j_i\|_\infty + C\varepsilon\|K\|_1\|J\|_\infty + Cq_0^2M^2T\varepsilon,$$
$$\|K\|_\infty \le C\|J\|_\infty + C\varepsilon\|J\|_1\|K\|_\infty + Cq_0^2M^2T\varepsilon.$$

The middle terms are absorbed by the established smallness thresholds. Thus both envelopes are bounded uniformly over existing $\mathcal{D}_S$.

Classical derivative bounds. The radial equation

$$\partial_v\left(\frac{\theta}{\lambda}\right)+\frac{\lambda}{r}\left(1-\left|\frac{\theta}{\lambda}\right|^2\right)\frac{\theta}{\lambda}=\frac{j}{1-\mu}$$

and its parallel-transported incoming counterpart, integrated from the axis, give

$$\left|\frac{\theta}{\lambda}\right|+\left|\frac{\xi}{\nu}\right|\le C_B r,\qquad \left|\partial_v\frac{\alpha}{\lambda}\right|+\left|D_u\frac{\beta}{\nu}\right|\le C_B.$$

Maxwell and the mass integral then give

$$|Q|\le C_B r^3,\quad |Q_u|+|Q_v|\le C_B r^2,\quad \mu\le C_B r^2,\quad |\mu_u|+|\mu_v|\le C_B r,\quad |\mathcal{C}|\le C_B r^2.$$

Consequently the right side of (3.34) is bounded by C_B. Initially $\mathcal{L}$ is bounded, since $\lambda|\theta/\lambda|^2/r = O_B(r)$ and the initial λ is constant. Integrating first for $\mathcal{L}$, then using its axis value for $\mathcal{N}$, bounds both by C_B. Subtract their quadratic corrections to obtain $|\lambda_v|+|\nu_u|\le C_B$. The mixed radial equation gives $|\lambda_u|+|\nu_v|\le C_B r$. The normalized scalar transport equations bound the other derivatives of α/λ and β/ν. The identities

$$\phi_v=\frac{\lambda}{r}\frac{\theta}{\lambda},\qquad \phi_u=\frac{\nu}{r}\frac{\xi}{\nu}-iq_0A_u\phi$$

bound the derivatives of ϕ. The radial equations just used, and the two wave transport equations, bound all four first derivatives of θ,ξ. The two mass constraints and Maxwell equations bound those of m,Q.

The five-term formula (3.64) also bounds $(A_u)_u$: its endpoint term vanishes, and the orders $Q=O_B(r^3)$, $Q_u=O_B(r^2)$, $\mu_u=O_B(r)$ make every integrand bounded. The potential constraint bounds $(A_u)_v$. Radial averaging gives the continuous center limits. Together with $r_u=\nu, r_v=\lambda$, these estimates supply a uniform bound B for the full C^1 norm on existing $\mathcal{D}_S$, including $\alpha/\lambda, \beta/\nu$ and the fixed initial data, with a positive margin.

Lemma 3.8 (A uniform positive-radius strip)**.** *Fix $B, c>0$ and a maximum side length T. Suppose a classical solution is known on the entering sides of $[0,U]\times[S,S+\delta]$, where $r\ge c$ on the entire entering side $v=S$ (equivalently at its endpoint (U,S), since $\nu<0$), its C^1 norm is at most B, and its geometric denominators have fixed positive lower bounds. For the strips used below, where the $u=0$ data come from one fixed C^1 initial cone, there is a $\delta>0$ uniform in $S,U\le T$ for which the characteristic integral equations have a classical solution with norm at most $2B$. One may enlarge B by a fixed factor before this assertion.*

Proof. On a compact neighborhood of the old fields with $r\ge c/2$ and fixed positive denominators, the first-order system has a uniform Lipschitz constant. As in rectangle uniqueness, integrate $r,\lambda,m,Q,\phi,\theta$ in u from $u=0$, and ν,ξ,A_u in v from $v=S$; the unused constraints propagate.

In the norm $\sup e^{-Ku}|\delta\mathbf{U}|$, a u integral has norm at most K^{-1} and a v integral norm at most δ. Choose K larger than four times the coefficient Lipschitz constant, then δ so that the second product is at most $1/4$. The reference $\mathbf{U}(u,S)+(v-S)\partial_v\mathbf{U}(u,S)$, corrected to match the prescribed $u=0$ data, is C^1: its mixed derivatives come from differentiating the first-order equations. Its first derivatives agree with the old trace at $v=S$. The value residual is $O_B(\delta)$; reducing δ by the fixed factor e^{-KT} keeps the iteration inside the nondegenerate ball.

Differentiate the integral equations once. The transverse derivatives solve linear integral equations with the same weighted bounds; use the norm of the values plus a sufficiently small fixed multiple

of the derivative norm to absorb the direct source terms. The coefficient change from $v = S$ is $O_B(\delta)$, and the initial transverse derivative changes by its modulus of continuity $\omega_i(\delta)$. Gronwall on $0 \le u \le T$ therefore bounds the derivative change by $C_B(\delta + \omega_i(\delta))$, uniformly in S. This tends to zero and gives the claimed $2B$ bound after reducing δ. The same estimate applies to $h = \phi + \theta/\lambda$ and its first derivatives, so $2B$ also controls the normalized datum required by the axis map. This proves uniform classical existence on the strip without second derivatives of the initial data. □

Filling the corner and including the terminal edge. Let S_* be the supremum of the constructed full triangles. It is positive by Lemma 3.7. Choose parameters in the following order. First fix the preceding full C^1 bound B. Next take a common axis Picard time $\tau > 0$ for later-cone data with C^1 norm at most $2B$ and the fixed strict small-data margins. Reduce τ at this point, if necessary, so that every old B-bounded solution restricted to such a short corner lies in the same Picard ball: its integral equations and their first v derivatives bound its distance from the reference pair by $C_B\tau$. Only after this reduction choose

$$0 < \eta < \min(S_*/4, \tau/8).$$

For $S < S_*$ sufficiently close to S_* append the slightly wider positive-radius strip

$$0 \le u \le S - \eta/2, \qquad S \le v \le S + \delta.$$

At its entering edge $r \ge 3\eta/16$. Lemma 3.8 supplies a common $\delta > 0$ and a $2B$ bound; reduce it so that $\delta < \min(\eta/4, \tau/8)$ and so that the data have the strict small-data margins required by the later-cone map. The variation added on a new interval of length δ is at most $2B\delta$. All these choices are independent of S near S_*.

The cone $u_0 = S - \eta$, from its known regular center to $v = S + \delta$, now has C^1 norm at most $2B$. Its length is $\eta + \delta < \tau$. The axis map therefore constructs its entire corner $u_0 \le u \le v \le S + \delta$. On $v \le S$ it agrees with the old classical solution by the uniqueness in the same local Picard class, ensured by the choice of τ before η. On the remaining overlap with the strip, apply Lemma 3.5 to

$$[S - \eta, S - \eta/2] \times [S, S + \delta].$$

Its two entering traces agree and its radius is positive. Thus the two constructions coincide there. Their union with the old triangle covers every point of $\mathcal{D}_{S+\delta}$. This contradicts $S_* < T$.

One final use of the same construction is necessary for a closed domain. Even if $S_* = T$, choose $S < T$ so close that $T - S < \delta$, and run the strip and corner construction capped at $v = T$. Only the prescribed initial data on $[0, T]$ are used. This includes the terminal null edge and the last axis point, completing the construction on the closed $\mathcal{D}_T$.

3.5.3 Threshold-preserving approximation and the strong AC limit

The estimates used in this passage are uniform on $\mathcal{D}_T$ and independent of the initial derivative supremum.

An approximation preserving both thresholds. Let $h = (\alpha/\lambda)(0, \cdot) \in \mathrm{AC}[0, T]$. Extend h' by zero to the real line, choose a nonnegative smooth approximate identity ρ_n of integral one, and put

$$h_n(v) = h(0) + \int_0^v (\rho_n * h')(s)\, \mathrm{d}s.$$

Then $h_n(0) = h(0)$, and Tonelli's theorem gives

$$\int_0^T |h_n'(v)|\,\mathrm{d}v \le \int_{\mathbb{R}} |\rho_n * h'| \le \int_0^T |h'(v)|\,\mathrm{d}v = \varepsilon.$$

Also $h_n' \to h'$ in $L^1[0,T]$, and $\|h_n - h\|_\infty \le \|h_n' - h'\|_1$. Thus $h_n \to h$ in $W^{1,1}$ without changing the vertex value or increasing the total variation. In particular $M_n = 1 + |h_n(0)| + 10\,\mathrm{TV}(h_n) \le M$, so both $\mathrm{TV}(h_n) \le \varepsilon_0$ and $|q_0|RM_n \le \delta_0$ hold even when the original inequalities are equalities. Apply the classical construction and its complete-triangle continuation to these data.

Strong derivative limits. Estimate (3.62) makes j_n Cauchy in $\int_0^T \sup_{u\le v} |j_n(u,v)|\,\mathrm{d}v$, and k_n Cauchy in the reflected norm. To construct common representatives, take a subsequence with summable consecutive differences and set

$$J_*(v) = \sup_{0\le u\le v} |j_1(u,v)| + \sum_{n=1}^{\infty} \sup_{0\le u\le v} |j_{n+1}(u,v) - j_n(u,v)|,$$
$$K_*(u) = \sup_{u\le v\le T} |k_1(u,v)| + \sum_{n=1}^{\infty} \sup_{u\le v\le T} |k_{n+1}(u,v) - k_n(u,v)|.$$

These envelopes are integrable. Outside one null set of labels, $j_n(\cdot, v)$ converges uniformly, including its axis endpoint, to a continuous representative; likewise for $k_n(u,\cdot)$. The limits are measurable, dominated by J_*, K_* on every section, and the full sequences converge in the envelope norms by the Cauchy estimate.

One may take the canonical envelopes $J_n(v) = \sup_{u\le v} |j_n(u,v)|$ and $K_n(u) = \sup_{v\ge u} |k_n(u,v)|$. The inequalities $|J_n - J_m| \le \sup_u |j_n - j_m|$ and its incoming analogue make these envelopes Cauchy in L^1. Consequently, if

$$\omega(a) = \sup_n \sup_{\substack{I\subset[0,T]\ \text{interval}\\ |I|\le a}} \int_I (J_n + K_n),$$

then $\omega(a) \to 0$ as $a \downarrow 0$. This is uniform integrability, not just a bound for their integrals.

No flux concentration at the axis. The radial moment and local Hardy estimates imply

$$\sup_{v-u\le a} (|\theta_n/\lambda_n| + |\xi_n/\nu_n|) \le C\omega(a),$$
$$\int_u^{\min(u+a,T)} \frac{\lambda_n}{r_n} |\theta_n/\lambda_n|\,\mathrm{d}v \le C \int_u^{\min(u+a,T)} J_n,$$
$$\int_{\max(0,v-a)}^{v} \frac{-\nu_n}{r_n} |\xi_n/\nu_n|\,\mathrm{d}u \le C \int_{\max(0,v-a)}^{v} K_n.$$

Multiplication by the uniform $C\varepsilon$ bound for the ratios proves that each corresponding quadratic flux has integral at most $C\varepsilon\omega(a)$. For the electric terms, the outgoing Maxwell constraint and zero central charge give, in radial coordinates,

$$|Q_n(u,r)| \le C|q_0|M \int_0^r s|\theta_n/\lambda_n|(u,s)\,\mathrm{d}s.$$

Hence $Q_n/r_n^2 \to 0$ uniformly at the axis, uniformly in n; the incoming Maxwell constraint gives the same oriented assertion.

The positive radial mass equation, in either orientation, has the form

$$\partial_r m_n = \frac{1-\mu_n}{2}|z_n|^2 + \frac{Q_n^2}{2r^2}, \qquad z_n = \theta_n/\lambda_n \text{ or } \xi_n/\nu_n.$$

Since $m_n(0) = 0$ and $0 \le 1-\mu_n \le 1$, integration and Tonelli yield

$$\int_0^a \frac{\mu_n(r)}{r}\,\mathrm{d}r \le \int_0^a \frac{|z_n(r)|^2}{r}\,\mathrm{d}r + \int_0^a \frac{Q_n(r)^2}{r^3}\,\mathrm{d}r.$$

This tends to zero uniformly over characteristic sections. Radial and coordinate strip widths are comparable in the fixed geometric box, so the estimates exclude scalar, mass and electric concentration at the axis.

Identifying derivatives and passing to each equation. The comparison estimate gives uniform convergence of the fields. Away from the axis, coefficients and their inverses converge uniformly; the nonconcentration estimates handle the remaining axis strips.

First, in

$$\partial_v(\alpha/\lambda)_n = \frac{j_n}{1-\mu_n} + \frac{\lambda_n}{r_n}|\theta_n/\lambda_n|^2(\theta_n/\lambda_n),$$

the first term converges strongly in L^1, uniformly over outgoing rows. The cubic term does so as well: its difference is bounded by the weighted scalar difference estimate and the geometric difference, or one may split at $v-u=a$ and use the quadratic nonconcentration bound times $C\varepsilon$ in the axis strip. The corresponding incoming equation uses k_n and the uniformly convergent connection $A_{u,n}$. The fundamental theorem of calculus on each closed section therefore identifies the limiting normalized fields as AC and identifies j, k almost everywhere with their renormalized derivatives.

The remaining equations pass as follows.

1. Uniform convergence passes $r_v = \lambda, r_u = \nu$, giving $r \in C^1$, and passes the radial averages identifying ϕ, θ, ξ. Linear Hardy nonconcentration, combined with uniform convergence off the axis, gives strong characteristic L^1 convergence of $\phi_v, D_u\phi$. Thus both Maxwell currents and the scalar transport sources converge strongly, with zero central charge.

2. The quadratic and electric bounds give strong convergence of the mass fluxes. Their radial averages control the weighted μ/r term, so the equations for μ and r_{uv} pass also on sections ending at the axis.

3. The Raychaudhuri fluxes are the controlled quadratic Hardy fluxes. Strong comparison of the geometric renormalizations gives λ_v, ν_u in characteristic L^1; the mixed equation gives $\lambda_u = \nu_v$. Hence $\lambda, \nu, \mu, \Omega^2 = -4\nu\lambda/(1-\mu)$ retain AC characteristic traces.

4. The potential constraint passes using the charge bound; (3.66), including the vanishing moving-axis term, passes its incoming derivative. Thus A_u is AC in both directions. Initial and axis values pass by uniform convergence and the uniform vanishing of the scalar ratios and μ.

The axis identity also passes in its proper trace sense:

$$\int_0^T |j_n(u,u) - j_m(u,u)|\,\mathrm{d}u \le \Delta_j^{n,m}, \qquad \frac{\mathrm{d}}{\mathrm{d}u}\phi_n(u,u) = j_n(u,u) = k_n(u,u).$$

Uniform convergence of the traces and strong L^1 convergence of their derivatives give the common AC axis value.

For two AC data sets, apply the classical comparison estimate to their threshold-preserving approximations and pass to the strong limits. This proves (3.62) and the ordinary, fixed-gauge characteristic bound

$$\sup_u \mathrm{TV}(\delta(\alpha/\lambda)(u,\cdot)) + \sup_v \mathrm{TV}(\delta(\beta/\nu)(\cdot,v)) \le C\Delta_0.$$

The incoming connection differences were included in (3.63). Uniqueness against arbitrary competitors follows from Proposition 3.6, without assuming that such competitors possess derivative envelopes.

3.6 The finite-cone restart used by extension arguments

Suppose an existing regular-center AC solution supplies a complete outgoing cone of radius R whose normalized datum has variation ε. In the gauge $A_v = 0$, relabeling this cone by $v = 2r$ leaves $\alpha/\lambda = \partial_r(r\phi)$ and its total variation unchanged. Relabel the incoming coordinate by the corresponding axis label and use its remaining electromagnetic gauge freedom to set $A_u|_\Gamma = 0$. This changes the initial datum only by one constant phase.

Fix $L > 1$. Extend α/λ constantly from radius R to radius LR, and reconstruct the cone fields from the radial constraints. This extension is AC and has precisely the original total variation. The constraints have unique solutions: Q is an explicit radial integral, the mass equation is linear in m, and ν, A_u are then explicit integrals. Consequently the reconstructed fields agree with the old ones on $0 \le r \le R$. Theorem 3.2 produces a solution on $0 \le u \le v \le 2LR$ provided

$$\varepsilon \le \varepsilon_0, \qquad |q_0|LR(1 + |\phi_{\mathrm{vertex}}| + 10\varepsilon) \le \delta_0.$$

Proposition 3.6 identifies it with the old solution on their common domain of dependence. The extension argument locates the proposed terminal point inside the enlarged triangle.

In particular, a sequence of complete cones with $R_n \to 0$, $\varepsilon_n \to 0$, and $R_n|\phi_{\mathrm{vertex},n}| \to 0$ satisfies the hypotheses for each fixed enlargement factor L. No uniform scalar-amplitude bound is needed beyond this scaled condition. The result concerns the AC subclass of BV; data with jumps require a separate axis formulation.

Preserving C^1 regularity in the first stage. If the selected outgoing datum is C^1, constant prolongation at its outer endpoint need not be C^1. Use instead a short taper of its radial derivative. Writing $H = \alpha/\lambda$ on that row, choose a smooth cutoff χ on $[0,\infty)$ with $\chi(0) = 1$ and $\chi = 0$ on $[1,\infty)$, and prescribe

$$\widetilde{H}(R+s) = H(R) + \partial_r H(R) \int_0^s \chi(t/\delta)\,\mathrm{d}t, \qquad 0 \le s \le (L-1)R.$$

This matches both the value and first derivative at R and is constant after $s = \delta$. The added variation is at most $\delta|\partial_r H(R)| \int_0^1 |\chi(t)|\,\mathrm{d}t$ and can be made arbitrarily small. Both smallness conditions must have a strict margin before this taper is applied. If its added variation is d, the new amplitude parameter is $M + 10d$, not M. Choose the taper width so that

$$\varepsilon + d < \varepsilon_0, \qquad |q_0|LR(M + 10d) < \delta_0, \qquad \delta < (L-1)R.$$

This is possible because $d \le \delta|\partial_r H(R)| \int_0^1 |\chi(t)|\,dt \to 0$ and both original inequalities are strict. The second condition is automatic when $q_0 = 0$. The C^1 construction with these updated parameters and overlap uniqueness give a C^1 continuation in the first stage.

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